Spearman's Footrule as a Measure of Disarray

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SUMMARY

Spearman's measure of disarray D is the sum of the absolute values of the difference between the ranks. We treat D as a metric on the set of permutations. The limiting mean, variance and normality are established. D is shown to be related to the metric I arising from Kendall's τ through the combinatorial inequality $I \leqslant D \leqslant 2I$.

Keywords: MEASURES OF ASSOCIATION; SPEARMAN'S FOOTRULE; KENDALL'S TAU; METRIC ON PERMUTATIONS; COMBINATORICS

1. INTRODUCTION

WIDELY used non-parametric measures of association such as Kendall's τ and Spearman's ρ lead to natural metrics on the set S_n of permutations of n letters. Let π and σ be elements of S_n. Four metrics which we consider are

$$D(\pi,\sigma) = \sum_{i=1}^{n} |\pi(i)-\sigma(i)|, \tag{1.1}$$

$$S(\pi,\sigma) = \sum_{i=1}^{n} \{\pi(i)-\sigma(i)\}^2, \tag{1.2}$$

$T(\pi,\sigma) =$ the minimum number of transpositions required to bring $\{\pi(1),\ldots,\pi(n)\}$ into the order $\{\sigma(1),\ldots,\sigma(n)\}$, (1.3)

$I(\pi,\sigma) =$ the minimum number of pairwise adjacent transpositions required to bring $\{\pi^{-1}(1),\ldots,\pi^{-1}(n)\}$ into the order $\{\sigma^{-1}(1),\ldots,\sigma^{-1}(n)\}$. Here π^{-1} and σ^{-1} are the permutations inverse to π and σ. (1.4)

In Section 2 we derive the mean, variance and limiting normality of $D(\pi,\sigma)$ when π and σ are chosen independently and uniformly from S_n. Similar, known, results for S, T and I are summarized in Table 1. In Section 3 we prove that $I+T \leqslant D \leqslant 2I$. The proof and subsequent discussion of boundary cases depends on combinatorial arguments of independent interest. In Section 4 we discuss the statistical implications of Sections 2 and 3 along with some open problems.

2. PROPERTIES OF THE FOUR METRICS

If d denotes one of the functions D, S, T or I, then it is easy to verify that d is a metric on S_n; that is:

$$d(\pi,\sigma) \geqslant 0 \text{ and } d(\pi,\sigma) = 0 \text{ if and only if } \pi = \sigma,$$

$$d(\pi,\sigma) = d(\sigma,\pi) \text{ and } d(\pi,\sigma) \leqslant d(\pi,\eta)+d(\eta,\sigma).$$

Further, all four metrics are *right invariant* in the sense that $d(\pi,\sigma) = d(\pi\eta,\sigma\eta)$ for $\eta \in S_n$. In particular $d(1,\pi) = d(1,\pi^{-1})$, where we write 1 for the identity permutation in S_n. We shall usually abbreviate $d(1,\pi)$ by $d(\pi)$. We consider more specific properties below.

$D(\pi,\sigma)$: The sum of the absolute values of the difference in the ranks is the metric associated with Spearman's footrule. Kendall (1970) discusses the footrule as a measure of association and dismisses it because of a lack of available statistical properties. We now prove Theorem 1.

Theorem 1. Let π and σ be permutations chosen independently and uniformly in S_n. Then, as $n\to\infty$

$$E\{D(\pi,\sigma)\} = \tfrac{1}{3}n^2+O(n), \tag{2.1}$$

$$\operatorname{var}\{D(\pi,\sigma)\} = \tfrac{2}{45}n^3+O(n^2) \tag{2.2}$$

and

$$D(\pi,\sigma), \text{ standardized by (2.1) and (2.2), is asymptotically normal.} \tag{2.3}$$

Proof. Using right invariance, the distribution of $D(\pi,\sigma)$ is the same as the distribution of $D(1,\pi)=D(\pi)$. Thus we may restrict our attention to $D(\pi)$.

We have

$$E\{D(\pi)\} = \sum_{i=1}^{n} n^{-1}\sum_{j=1}^{n}|i-j| = \tfrac{1}{3}n^2+O(n)$$

which proves (2.1). The proof of (2.2) is lengthy and we sketch only the details. Consider

$$\begin{aligned}\operatorname{var}\{D(\pi)\} &= \sum_{i=1}^{n}\operatorname{var}|i-\pi(i)|+2\sum_{i<j}\operatorname{cov}(|i-\pi(i)|,|j-\pi(j)|)\\ &= \tfrac{1}{20}n^3+O(n^2)+2\sum_{i<j}\operatorname{cov}(|i-\pi(i)|,|j-\pi(j)|).\end{aligned} \tag{2.4}$$

To simplify the evaluation of the covraiance sum note that for any bounded functions f and g,

$$\begin{aligned}E[f\{\pi(i)\}g\{\pi(j)\}] &= \{n(n-1)\}^{-1}\sum_{l\neq k} f(l)g(k)\\ &= \{n(n-1)^{-1}\}E[f\{\pi(1)\}]E[g\{\pi(1)\}]-(n-1)^{-1}E[f\{\pi(1)\}g\{\pi(1)\}]\\ &= E[f\{\pi(1)\}]E[g(\pi(1)\}]-(n-1)^{-1}\operatorname{cov}[f\{\pi(1),g\{\pi(1)\}].\end{aligned}$$

In particular

$$\operatorname{cov}(|i-\pi(i)|,|j-\pi(j)|) = -(n-1)^{-1}\operatorname{cov}(|i-\pi(1)|,|j-\pi(1)|). \tag{2.5}$$

Consider

$$\begin{aligned}\operatorname{cov}(|i-\pi(1)|,|j-\pi(1)|) &= n^{-1}\sum_{l=1}^{n}|i-l||j-l|\\ &\quad -(4n^2)^{-1}\{i(i-1)+n(n-i-1)\}\{j(j-1)+(n-j)(n-j-1)\}\\ &= f_1(i,j)+f_2(i,j).\end{aligned} \tag{2.6}$$

Summing $f_1(i,j)$ in $i<j$ for fixed j and then in j yields

$$\sum_{i<j} f_1(i,j) = \tfrac{7}{120}n^4+O(n^3). \tag{2.7}$$

Similarly,

$$\sum_{i<j} f_2(i,j) = -\tfrac{1}{18}n^4+O(n^3). \tag{2.8}$$

Then (2.8) and (2.7), along with (2.6) and (2.5), show that

$$\sum_{i<j}\operatorname{cov}(|i-\pi(i)|,|j-\pi(j)|) = -\tfrac{1}{360}n^3+O(n^2).$$

Using this in (2.4) completes the proof of (2.2).

To prove (2.3) we use Hoeffding's (1951) combinatorial central limit theorem. This states that if $\{a_{ij}^n\}$, $i,j=1,\ldots,n$, is a sequence of square arrays and if $W_n=\sum_{i=1}^{n}a_{i\pi(i)}^n$, where π is a random permutation in S_n, then subject to growth conditions on a_{ij}^n, W_n is asymptotically normal. Using (2.2) we readily verify the sufficient condition given in equation (12) of Hoeffding (1951) for the array

$$a_{ij}^n = |i-j|, \quad i,j = 1, \ldots, n.$$

This completes the proof of Theorem 1.

$S(\pi, \sigma)$: The sum of squares of the difference in the ranks is the metric associated with Spearman's coefficient of association ρ. All of the results listed in Table 1 may be found in Kendall (1970).

$T(\pi, \sigma)$: This seemingly natural metric has not received much attention in the statistical literature. Note first that if ψ and η are in S_n, we have

$$T(\pi, \sigma) = T(\psi\pi\eta, \psi\sigma\eta) = T(1, \pi^{-1}\sigma) = T(\pi^{-1}\sigma).$$

A result due to Cayley (1849) states that $T(\pi^{-1}\sigma) = n - C(\pi^{-1}\sigma)$ where $C(\eta)$ is the *number of cycles* in η. This gives an easy way to compute $T(\pi, \sigma)$. Feller (1968, p. 256) gives a representation of the number of cycles in a random permutation as a sum of independent random variables. This representation and standard theorems from probability theory easily imply the limiting normality of T as well as the results listed in Table 1.

TABLE 1

Properties of the metrics

	Max	*Mean*	*Variance*
D	$[\frac{1}{2}n^2]$	$\frac{1}{3}n^2$	$\frac{2}{45}n^3$
S	$\frac{1}{3}(n^3-n)$	$\frac{1}{6}n^3$	$\frac{1}{36}n^5$
T	$n-1$	$n-\log n$	$\log n$
I	$\frac{1}{2}(n^2-n)$	$\frac{1}{4}n^2$	$\frac{1}{36}n^3$

$I(\pi, \sigma)$: This metric arises from Kendall's measure of association τ. It is thoroughly discussed in Kendall (1970). A simple approach to the analysis of the stochastic properties of this metric is as follows: A permutation η is said to *have an inversion at* (k, l) if $k < l$ and $\eta(k) > \eta(l)$. A standard result (see, for example, Knuth, 1973, Section 5.1.1) is that $I(\pi, \sigma) = I(1, \pi\sigma^{-1})$ = the number of inversions in $\pi\sigma^{-1}$. Feller (1968, p. 256) gives a representation for the number of inversions in a random permutation as a sum of independent random variables. From this the limiting normality, as well as the results listed in Table 1, follow easily.

In Table 1, only the leading terms of the mean and variance are indicated, where $[x]$ denotes the integral part of x.

3. SOME INEQUALITIES INVOLVING $D(\pi, \sigma)$

There are important relations between the various metrics which generally take the form of inequalities which must be satisfied. For example, Kendall (1970) discusses the Durbin–Stuart inequality, which in our notation is

$$S(\pi, \sigma) \geqslant \tfrac{4}{3} I(\pi, \sigma)\,[1 + \{I(\pi, \sigma)/n\}].$$

We now prove the next theorem.

Theorem 2.

$$I(\pi, \sigma) + T(\pi, \sigma) \leqslant D(\pi, \sigma) \leqslant 2I(\pi, \sigma). \tag{3.1}$$

Proof. Since all the quantities in (3.1) are right invariant it is sufficient to show that

$$I(\pi) + T(\pi) \leqslant D(\pi) \leqslant 2I(\pi). \tag{3.2}$$

We prove the right-hand inequality first.

$I(\pi) = I(\pi^{-1})$ is the smallest number of pairwise adjacent transpositions required to being π to the identity. Let x_i, $1 \leqslant i \leqslant I(\pi)$, be a sequence of integers which indexes a sequence of transpositions which transform 1 to π_1 to π_2 to ... to π. The ith transposition transforms π_i to π_{i+1} by interchanging $\pi_i(x_i)$ and $\pi_i(x_i+1)$. We may assume that the sequence x_i is chosen so that $\pi_i(x_i) < \pi_i(x_i+1)$. Consider the difference

$$\Delta_{i+1} = D(\pi_{i+1}) - D(\pi_i) = |x_i - \pi_i(x_i+1)| + |x_i+1-\pi_i(x_i)| - |x_i - \pi_i(x_i)| - |x_i+1-\pi_i(x_i+1)|.$$

There are three possibilities:

Case 1. $\pi_i(x_i+1) \leqslant x_i$. Then $\pi_i(x_i) < x_i$ and $\Delta_{i+1} = 0$.

Case 2. $\pi_i(x_i) \geqslant x_i+1$. Then $\Delta_{i+1} = 0$.

Case 3. $\pi_i(x_i+1) \geqslant x_i+1$, $\pi_i(x_i) \leqslant x_i$. Then $\Delta_{i+1} = 2$.

Thus $D(\pi) = \sum_{i=1}^{I(\pi)} \Delta_i \leqslant 2I(\pi)$ as desired.

To prove the left-hand side of the inequality (3.2) we need some more notation.

Denote the inversion $\pi(k) > \pi(l)$ with $k < l$ by $[k; l]$. Let us call $[k; l]$ a *Type I* inversion if $\pi(k) \geqslant l$ and a *Type II* inversion if $\pi(k) \leqslant l$. Thus, every inversion of π is either a Type I or a Type II inversion and some inversions may be of both types.

For a fixed k, if $[k; y]$ is a Type I inversion then we must have

$$k < y \leqslant \pi(k).$$

Therefore, denoting the number of elements in a finite set A by $|A|$,

$$|\{y\colon [k; y] \text{ is a Type I inversion}\}| \leqslant \pi(k) - k. \tag{3.3}$$

Similarly, if $[x; l]$ is a Type II inversion then $\pi(l) < \pi(x) \leqslant l$ and so

$$|\{x\colon [x; l] \text{ is a Type II inversion}\}| \leqslant l - \pi(l). \tag{3.4}$$

Thus, it follows that

$$D(\pi) = \sum_k |k - \pi(k)| = \sum_{\pi(k) \geqslant k} (\pi(k) - k) + \sum_{\pi(l) \leqslant l} (l - \pi(l))$$

$$\geqslant \text{the number of Type I inversions} + \text{the number of Type II inversions} \geqslant I(\pi). \tag{3.5}$$

In order to prove (3.2), we have to examine (3.5) more carefully. In particular we must see how much was given away at each of the two inequalities in (3.5).

In the first place, suppose for some integer y,

$$k < y \leqslant \pi(k) < \pi(y). \tag{3.6}$$

Such a y is counted by the sum $\sum(\pi(k) - k)$, summing over $\pi(k) \geqslant k$, but is not counted in the sum of the numbers of Type I and Type II inversions since $[k; y]$ is not an inversion.

Similarly, if x satisfies

$$\pi(x) < \pi(l) \leqslant x < l \tag{3.7}$$

then x is counted by the sum $\sum(l - \pi(l))$, summing from $\pi(l) \leqslant l$, but $[x; l]$ is not an inversion. Thus, for each such x and y we get a contribution of 1 to $D(\pi) - I(\pi)$.

The other place in (3.5) where we have a potential gain of $D(\pi)$ over $I(\pi)$ is the last inequality. Here, we gain 1 for each inversion of π which is both Type I and Type II, i.e. for each x having

$$x < \pi(x), \quad \pi^2(x) < \pi(x). \tag{3.8}$$

Let us split (3.6) and (3.7) into two pieces each:

$$k<\pi(k)<\pi^2(k) \quad (\text{with } y=\pi(k)), \tag{3.6$'$}$$

$$k<y<\pi(k)<\pi(y), \tag{3.6$''$}$$

$$\pi^2(l)<\pi(l)<l \quad (\text{with } x=\pi(l)), \tag{3.7$'$}$$

$$\pi(x)<\pi(l)<x<l. \tag{3.7$''$}$$

Let us now restrict our attention to a cycle $C=(c_1,c_2,\ldots,c_m)$ of π of length $|C|=m$. Thus, $\pi(c_k)=c_{k+1}$ for $1\leqslant k<m$, and $\pi(c_m)=c_1$. We call an element or pair of elements of C satisfying (3.6′), (3.6″), (3.7′), (3.7″) or (3.8) a *critical configuration* of C.

We now show that C has at least $m-1$ critical configurations. We proceed by induction on m. For $m=1$ there is nothing to prove. For $m=2$ it is immediate since in this case (3.8) clearly holds for the smaller element of C. Assume for some $m>2$ that the claim is valid for all values $<m$ and suppose C is an m-cycle which has at most $m-2$ critical configurations. There are several cases:

Case 1. Suppose $x\in C$ satisfies (3.6′), i.e. $x<\pi(x)<\pi^2(x)$. Form the $(m-1)$-cycle C' by removing x from C. It is not hard to check that we have reduced the total number of critical configurations by at least one so that C' has at most $m-3$ critical configurations, which is a contradiction to the induction hypothesis.

Case 2. Similarly, if $x\in C$ satisfies (3.7′) then by deleting it from C, we form an $(m-1)$-cycle C' having at most $m-3$ critical configurations, which is impossible.

Case 3. Suppose there exists $x\in C$ satisfying

$$x<\pi^2(x)<\pi(x)<\pi^3(x).$$

Thus, C has a critical configuration satisfying (3.6″) with $k=x$ and $y=\pi^2(x)$. Form the $(m-2)$-cycle C' by deleting $\pi(x)$ and $\pi^2(x)$ from C. It is now not difficult to see that the total number of critical configurations remaining is at most $m-4$ (note that x also satisfies (3.8)), which contradicts the induction hypothesis.

Case 4. A similar argument applies if some $y\in C$ satisfies

$$y>\pi^2(y)>\pi(y)>\pi^3(y).$$

Therefore, we may assume that Cases 1–4 do *not* occur in C. However, if c denotes the *least* element of C then this assumption forces $\pi^{k+2}(c)$ always to be strictly between $\pi^k(c)$ and $\pi^{k+1}(c)$ for any $k\geqslant 0$. This is clearly impossible (since C is a cycle) and the induction is complete.

Thus, each m-cycle of π contributes at least $m-1$ to the difference $D(\pi)-I(\pi)$. Therefore

$$D(\pi)-I(\pi)\geqslant \sum_{C=\text{cycle of }\pi}(|C|-1)=\sum_C|C|-C(\pi)=n-C(\pi)$$

completing the proof of Theorem 2.

As a numerical example, Table 2 lists the values of the four metrics when $n=4$.

Remarks

1. An immediate deduction from Theorem 2 is the bound $C(\pi)+I(\pi)\geqslant n$.
2. We can easily characterize the permutations $\pi\in S_n$ where $D(\pi)$ takes on its maximum value. For n even, $D(\pi)\leqslant\frac{1}{2}n^2$ with equality if and only if $\pi(i)>\frac{1}{2}n$ for $i=1,2,\ldots,\frac{1}{2}n$. Thus $D(\pi)$ takes on its maximum value for $((\frac{1}{2}n)!)^2$ choices of π. A similar bound holds for odd n. This skewness partially explains why the mean of $D(\pi)$ is larger than $\frac{1}{4}n^2$.
3. It is natural to consider when equality is attained in Theorem 2. The example in Table 2 shows equality of the lower bound in 23 out of 24 cases. For larger n, consideration of the mean values shows that equality can occur for at most $o(n!)$ elements of S_n. We can characterize the elements $\pi\in S_n$ for which equality occurs in the upper bound. Let us say that π has a 3-*inversion* if there exist $i<j<k$ with $\pi(i)>\pi(j)>\pi(k)$. It can be shown that

$D(\pi) = 2I(\pi)$ if and only if π has no 3-inversions. Knuth (1973, Section 5.1.4) notes that the number of permutations in S_n with no 3-inversions is exactly the well-known Catalan number

$$\frac{1}{2n+1}\binom{2n+1}{n}.$$

4. The number of permutations π satisfying $I(\pi) = T(\pi)$ is easily seen to be the Fibonacci number F_{2n-1} defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. This shows that simultaneous equality of both bounds in Theorem 2 holds exponentially often.

TABLE 2

Values of the four metrics when $n = 4$

π	*Cycles*	$T(\pi)$	$I(\pi)$	$D(\pi)$	$S(\pi)$
(1234)	(1) (2) (3) (4)	0	0	0	0
(1243)	(1) (2) (34)	1	1	2	2
(1324)	(1) (23) (4)	1	1	2	2
(1342)	(1) (234)	2	2	4	6
(1423)	(1) (243)	2	2	4	6
(1432)	(1) (24) (3)	1	3	4	8
(2134)	(12) (3) (4)	1	1	2	2
(2143)	(12) (34)	2	2	4	4
(2314)	(123) (4)	2	2	4	6
(2341)	(1234)	3	3	6	12
(2413)	(1243)	3	3	6	10
(2431)	(124) (3)	2	4	6	14
(3124)	(132) (4)	2	2	4	6
(3142)	(1342)	3	3	6	10
(3214)	(13) (2) (4)	1	3	4	8
(3241)	(134) (2)	2	4	6	14
(3412)	(13) (24)	2	4	8	16
(3421)	(1324)	3	5	8	18
(4123)	(1432)	3	3	6	12
(4132)	(142) (3)	2	4	6	14
(4213)	(143) (2)	2	4	6	14
(4231)	(14) (2) (3)	1	5	6	18
(4312)	(1423)	3	5	8	18
(4321)	(14) (23)	2	6	8	20

4. INTERPRETATION OF RESULTS AND SOME OPEN PROBLEMS

Statisticians most often normalize metrics so that they have the properties of a correlation coefficient. The translation is straightforward: if d is a metric on S_n and its maximum value is m, define a measure of association by $R(\pi, \sigma) = 1 - \{2d(\pi, \sigma)/m\}$. We note in this connection the confusion on pp. 32–33 in Kendall (1970) concerning Spearman's footrule. It is clear that the proper choice of m for the footrule is $[\frac{1}{2}n^2]$. Knuth (1973) works with I and T directly as measures of disarray.

An example from Kendall (1970) will help clarify a discussion of invariance. Consider a class of ten students ranked in mathematics and music:

Pupil	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
Mathematics (π)	7	4	3	10	6	2	9	8	1	5
Music (σ)	5	7	3	10	1	9	6	2	8	4

A typical use of the metrics we suggest would be to compute the distance between the permutations π and σ as a measure of association. Right invariance here refers to invariance under changes in the labels a to j of the students. Right invariance is compelling in so many situations that we have forced all of the metrics to be right invariant. This accounts for the at first unnatural appearance of the definition of $I(\pi, \sigma)$. Left invariance corresponds to relabelling the ranks; as an example, it could be that 10 means the worst ranking and 1 the best, instead of the opposite relation. Among the metrics we consider, T is left invariant as well as right invariant.

The results in Table 1 suggest that T is unsuitable for general use, having very small variance about a mean very close to its maximum value. Of the three remaining matrics S has the largest range with corresponding greater variability. S has the natural interpretation of the Euclidean distance between the vectors π and σ regarded as points on the surface of a sphere in n dimensional space. I and D seem roughly similar, D seems somewhat easier to interpret directly while I has the advantage of having its distribution tabulated for small sample sizes (cf. Kendall, 1970). The inequalities in Section 3 also suggest that the differenc between I and N is not very great.

The quantities I and T arise in the analysis of sorting algorithms. In particular Knuth (1973, p. 141) considers the quantity $B(\pi)$ = the number of right to left maxima in π. Knuth (1973, p. 157) shows that in our notation $2B(\pi) \leqslant I(\pi)+T(\pi) \leqslant [\frac{1}{2}n^2]$.

Use of Theorem 2 actually gives the stronger inequality $2B(\pi) \leqslant D(\pi)$. The quantity $D(\pi)$ also appears in Problem 5.2.1–7 of Knuth (1973) in connection with another sorting algorithm.

We conclude by listing some open problems.

1. Find inequalities relating $D(1,\pi)$ to $S(1,\pi)$. A referee has pointed out the bound $S(\pi)/n-1 \leqslant D(\pi) \leqslant \min[S(\pi), \{nS(\pi)\}^{\frac{1}{2}}]$. This follows from $|x| \leqslant x^2 \leqslant (n-1)|x|$ for x an integer with $|x| \leqslant n-1$ and the Cauchy–Schwartz inequality. It is not hard to see that $D(\pi) = S(\pi)$ for exactly F_{n+1} permutations $\pi \in S_n$ (cf. Remark 4).
2. Find some reasonable two-sided invariant metrics.
3. Characterize the permutations in S_n where equality of the lower bound of Theorem 2 holds.

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