1 Outline

Agenda: Image formation

1. Image formation
2. Relation between object and image
3. Effects of diffraction

Last Time: We studied how to model the effect lenses have on propagating light. If the lenses are thin, we can assume the wave field only experiences a phase shift as it traverses the lens. The shift is proportional to the thickness of the lens, which we calculated in the case of circular, doubly convex lenses. Performing a paraxial approximation, we deduced that the lens introduces a quadratic phase shift. Furthermore, the properties of the lens can be summarized by a single quantity, the focal length. Consequently, using a lens allows us to observe the Fourier transform of the incoming wave, modulo modulation by a quadratic phase shift, even though we are in the Fresnel diffraction regime. Under certain assumptions, this also shows that if we observe a wave field at focal distance before the lens, then we can observe the Fourier transform of the field (up to a multiplicative factor) a focal length after the lens.

2 Image formation

Although we can form an image with a 4-\(f\) system, this arrangement has some drawbacks. In particular, the two lenses in the system must be aligned, and an additional distance of 2\(f\) is required. Now we discuss how one can image an object with a single lens.

The optical system can be represented as an integral transform with kernel \(h\), so that

\[
  u(\mathbf{x}_0) = \int u_0(\mathbf{x}) h(\mathbf{x}_0, \mathbf{x}) d\mathbf{x}, \quad (1)
\]

and, ideally, one would like to have

\[
  h(\mathbf{x}_0, \mathbf{x}) = A\delta \left( \mathbf{x}_0 - \frac{1}{M} \mathbf{x} \right),
\]
where $A$ is called the **amplitude factor**, and $M$ is the **magnification factor**. The constant $M$ may be negative, as the image could be inverted by the system. Can we achieve this in practice? The short answer is no. But we will see where are the sources of errors, and how can we still obtain a good approximation of such a system.

Using Fresnel’s diffraction formula for an optical system with a single lens (see Fig. 1), we can deduce the expression

$$u(x_0) = -\frac{1}{\lambda^2 z_1 z_2} e^{ik(z_1 + z_2 + n\Delta_0)} \iint u_0(x') P(x) e^{-ik\|x\|^2} e^{ik\|x - x'\|^2} dxd' dxd'',$$

where $P$ is the pupil function. The presence of $P$ means we are not neglecting the finite extent of the lens aperture. Suppose we had a bright spot on the focal plane behind the lens, which we model as $u_0 = \delta_{x_s}$. Then a formal computation shows that

$$u(x_0) = -\frac{1}{\lambda^2 z_1 z_2} e^{ik(z_1 + z_2 + n\Delta_0)} \int P(x) e^{-ik\|x\|^2} e^{ik\|x - x'\|^2} e^{ik\|x - x_0\|^2} dx$$

meaning a bright spot appears at $x_s$ as the pattern given by $\psi$, modulo a factor. Intuitively, the optical system will be good if this pattern is concentrated around $x_s$. As in Lecture 17, the phase term in the integral can be written as

$$\frac{k}{2} \left[ -\left( \frac{1}{f} - \frac{1}{z_1} - \frac{1}{z_2} \right) \|x\|^2 - 2 \left\langle x, \frac{1}{z_1} x_s + \frac{1}{z_2} x_0 \right\rangle + \frac{1}{z_1} \|x_s\|^2 + \frac{1}{z_2} \|x_0\|^2 \right].$$

**Figure 1:** Optical system used to form an image. In practical settings, $z_1$ and $z_2$ should satisfy the lens law (4).

We now use the **thin lens equation**, or **lens maker’s formula**:

$$\frac{1}{f} = \frac{1}{z_1} + \frac{1}{z_2},$$

This identity is easily seen using geometric optics. Examine Fig. 2a, which illustrates the result of imaging through a lens. In Fig. 2b, the blue and green triangles are similar; in particular,

$$\frac{z_2 - f}{f} = \frac{h_2}{h_1}.$$
Now, the red and green triangles in Fig. 2c are also similar, giving
\[ \frac{h_2}{h_1} = \frac{z_2}{z_1}. \]

Hence
\[ \frac{z_2 - f}{f} = \frac{z_2}{z_1}. \]

Rearranging terms and dividing by \( z_2 \), we have exactly (4).

From the thin lens equation, the quadratic term in (3) disappears, and the phase becomes
\[ \frac{k}{2} \left[ -2 \left\langle x, \frac{1}{z_1} x + \frac{1}{z_2} \xi_0 \right\rangle + \frac{1}{z_1} \| x \|^2 + \frac{1}{z_2} \| \xi_0 \|^2 \right]. \]

Now the integral in (2) can be computed, giving
\[ \psi(x_0, x_s) = \int P(x) e^{-i k \left\langle x, \frac{1}{z_1} x + \frac{1}{z_2} \xi_0 \right\rangle} e^{i k \| x \|^2} e^{i k \| \xi_0 \|^2} d\xi \]
\[ = e^{\frac{i k}{z_1} \| x_s \|^2} e^{\frac{i k}{z_2} \| \xi_0 \|^2} \tilde{P} \left( \frac{k}{z_2} \left( \frac{z_2}{z_1} x_s + \xi_0 \right) \right). \]

So we let
\[ M = -\frac{z_2}{z_1} \]
be the magnification factor, and conclude
\[ h(x_0, x_s) = -\frac{1}{\lambda^2 z_1 z_2} e^{i k (z_1 + z_2 + n \Delta_0)} e^{\frac{i k}{z_1} \| x_s \|^2} e^{\frac{i k}{z_2} \| \xi_0 \|^2} \tilde{P} \left( \frac{k}{z_2} (x_0 - M x_s) \right). \]

We see that the pattern obtained from a bright spot is determined by the Fourier transform of the pupil function. Furthermore, the brightest spot is located at the point \( x_0 = M x_s \). (Can you see why? Think about the Fourier transform of an indicator function.) The magnification and amplitude factors are determined by the specific values of \( z_1 \) and \( z_2 \) in the optical system. The quadratic phase term in \( x_0 \) can be neglected, as only the intensity is measured. However, the quadratic phase term in \( x_s \) cannot be neglected. The main issue is that this term can introduce significant phase shifts when we try to image. In summary, we have in (1)
\[ u(x_0) = -\frac{1}{\lambda^2 z_1 z_2} e^{i k (z_1 + z_2 + n \Delta_0)} e^{\frac{i k}{z_1} \| x_s \|^2} \int u_0(x') e^{\frac{i k}{z_2} \| x' \|^2} \tilde{P} \left( \frac{k}{z_2} (x_0 - M x') \right) d x'. \quad (5) \]

We can eliminate the effect of the modulation by a quadratic phase factor in two ways:

1. We can place a lens just in front of the wave field on the source plane. Note the focal length should be equal to \( z_1 \).
2. If we assume the source is well-localized in such a way that
\[ \frac{k}{2 z_1} \| x' \|^2 \ll \pi, \]
then the effects of the modulation by a quadratic phase term can simply be neglected.
Figure 2: Diagrams demonstrating the thin lens equation \((4)\) under the model of geometric optics. The object to be imaged is represented by the red arrow on the left, with height \(h_1\) and at distance \(z_1\) from the lens. The image is represented by the green arrow on the right, with height \(h_2\), at distance \(z_2\) from the lens. In (a), a light ray (orange) is refracted by the lens to pass through the focal point (blue). Its intersection with the corresponding straight-line ray determines the location of the image.
Suppose we follow the second case. Using the convolution theorem and Parseval-Plancherel identity, we can instead write the integral in (5) as

\[
\int u_0(x') \hat{P} \left( \frac{k}{z_2} (x_0 - M x') \right) d'x' = \frac{z_2^2}{k_2} \int u_0 \left( \frac{1}{M} x' \right) \hat{P} \left( \frac{k}{z_2} (x_0 - x') \right) d'x' = \frac{z_2^2}{k_2} \int \hat{u}_0(\omega) \hat{P}(\omega) e^{i\langle \omega, x_0 \rangle} d\omega,
\]

where \(\omega\) represents the conjugate variable to \(x\), and 

\[
\tilde{u}_0(x') = u_0 \left( \frac{1}{M} x' \right) \quad \text{and} \quad \tilde{P}(\omega) = P \left( -\frac{z_2}{k} \omega \right).
\]

Therefore, the intensity on the screen is

\[
I(x_0) = |u(x_0)|^2 = \frac{1}{(2\pi)^4} \left( \frac{1}{M} \right)^2 \left| \int \hat{u}_0(\omega) \hat{P}(\omega) e^{i\langle \omega, x_0 \rangle} d\omega \right|^2.
\]

If the wavelength tends to zero, that is, \(\lambda \to 0\), then \(\tilde{P} \to 1\) by construction. The integral becomes an inverse Fourier transform, and thus

\[
I(x_0) \to \frac{1}{M^2} |\tilde{u}_0(x_0)|^2 = \left| \frac{1}{M} u_0 \left( \frac{1}{M} x_0 \right) \right|^2.
\]

When the effects of the modulation by a quadratic phase factor cannot be neglected, we obtain a similar expression for

\[
\tilde{u}_0(x') = u_0 \left( \frac{1}{M} x' \right) e^{i \frac{ik}{z_2} \|x'\|^2}.
\]

This shows we recover geometric optics from wave optics, as the wavelength goes to zero. Indeed, geometric optics appear as an asymptotic limit of wave optics, failing to account for diffraction effects. In fact, neglecting phase terms, we can write

\[
u(x_0) \propto \int \frac{1}{|M|} u_0 \left( \frac{1}{M} x \right) h(x_0 - x) d x,
\]

with 

\[
h(x) = \frac{1}{(2\pi)^2} \left( \frac{k}{z_2} \right)^2 \hat{P} \left( \frac{k}{z_2} x \right).
\]

In the case of a circular lens of radius \(a\), we have\(^\text{1}\) the point spread function

\[
\hat{P}(x) = a \frac{J_1(a \|x\|)}{\|x\|},
\]

Therefore, we are convolving with a kernel that approximately averages the image over circular regions of radius

\[
r \approx 1.22 \frac{\lambda}{d z_2},
\]

where \(d\) is the diameter of the aperture. A mock example can be seen in Fig.\(^\text{3}\). Recall that the central bright spot of the PSF \(\hat{P}(x)\) is called the Airy disk.

\(^{1}\)See section 6.2 of the Lecture 16 notes.
The convolution clearly has a blurring effect, explained by the averaging process. But we can also understand this as the result of a low-pass filter. Indeed, in the spatial domain, $\hat{P}$ corresponds to $P$, which is the indicator function of a circular lens aperture. Considered as a transfer function, $P$ eliminates all frequency content for $\|\omega\| > a$. Consequently, we observe a true low-pass version of the original image.

Figure 3: The Lena image convolved with the point spread function of a circular aperture. The result is a low-pass filtered version of the original image, corresponding to the local averaging performed by the convolution. The Airy disk, whose width is determined by the first zero of the Bessel function $J_1$, determines the region of greatest contribution to each local average.