1 Outline

**Agenda:** Fourier Integrals (continued)

1. Fourier inversion formula
2. Convolution theorem
3. Properties of the Fourier transform

**Last Time:** We studied time-invariant operators (TIOs) and showed how these can be represented as convolution with a fixed function $h$, which is the response of the operator to the impulse $\delta$. Their eigenfunctions are complex exponentials $e^{i\omega t}$, with eigenvalues given by the transfer function $\hat{h}(\omega)$. This motivated the definition of the Fourier transform as the map $h \mapsto \hat{h}$ given by

\[
\hat{h}(\omega) = \int h(t)e^{-i\omega t} dt,
\]
and defined for summable functions $h \in L^1(\mathbb{R})$. For example,

\[
\begin{align*}
f(t) &= \mathbb{I}_{\{-1/2 \leq t \leq 1/2\}} \quad \mapsto \quad \hat{f}(\omega) = \text{sinc}(\omega/2\pi) \\
f(t) &= (1 - |t|)\mathbb{I}_{\{-1 \leq t \leq 1\}} \quad \mapsto \quad \hat{f}(\omega) = \text{sinc}^2(\omega/2\pi) \\
f(t) &= \frac{1}{\sqrt{2\pi}}e^{-t^2/2} \quad \mapsto \quad \hat{f}(\omega) = e^{-\omega^2/2}
\end{align*}
\]

2 Fourier inversion formula

In signal recovery, it is essential that Fourier transforms are unique (i.e. the Fourier transform is injective). This fact is not clear a priori, but it follows from the next result:

**Theorem 1** (Fourier inversion formula). Suppose $f, \hat{f} \in L^1(\mathbb{R})$. Then

\[
f(t) = \frac{1}{2\pi} \int \hat{f}(\omega)e^{i\omega t} d\omega. \quad (1)
\]

This is an extremely important result. It shows we can write $f$ (under the hypotheses of the theorem) as a superposition of complex exponentials. Additionally, notice that the inverse formula looks very similar to the expression for the Fourier transform (modulo a factor $1/2\pi$ and the conjugation of the exponential in the integrand).
Proof. We make the additional assumption that \( f \) and \( \hat{f} \) are continuous (written \( f, \hat{f} \in C^0(\mathbb{R}) \)), a fact we justify in the remark below. Writing the Fourier transform explicitly, the expression for the inverse reads

\[
f(t) = \frac{1}{2\pi} \int \left( \int f(u) e^{-iu\omega} \, du \right) e^{i\omega t} \, d\omega.
\]  

(2)

It seems tempting to exchange the order of integration in the above expression. However, we cannot do this, as the hypotheses of Fubini’s theorem are not satisfied (that is, \( f(u) e^{i\omega(t-u)} \) is not in \( L^1(\mathbb{R} \times \mathbb{R}) \)). We use the following trick. Define

\[
I_\varepsilon(t) = \frac{1}{2\pi} \int \int f(u) e^{i\omega(t-u)} e^{-\varepsilon^2 \omega^2 / 2} \, du \, d\omega,
\]

which one can reasonably see as an approximation to the RHS of (2), as \( \varepsilon \downarrow 0 \). In this case the integrand is summable and we can invoke Fubini’s theorem to change the order of integration. If we integrate over \( u \) first we have

\[
I_\varepsilon(t) = \frac{1}{2\pi} \int \hat{f}(\omega) e^{i\omega t} e^{-\varepsilon^2 \omega^2 / 2} \, d\omega \xrightarrow[\varepsilon \downarrow 0]{\text{as}} \frac{1}{2\pi} \int \hat{f}(\omega) e^{i\omega t} \, d\omega.
\]

The limit taken as \( \varepsilon \downarrow 0 \) is justified by the Dominated Convergence Theorem (DCT): the integrand is dominated by \( |\hat{f}(\omega)| \), which is summable by hypothesis. On the other hand, integrating over \( \omega \) first gives

\[
I_\varepsilon(t) = \int f(u) g_\varepsilon(t-u) \, du, \quad \text{where} \quad g_\varepsilon(u) = \frac{1}{\varepsilon} g\left(\frac{u}{\varepsilon}\right)
\]

and \( g(t) = (\sqrt{2\pi})^{-1} e^{-t^2/2} \). Note that \( g_\varepsilon \) is positive, has integral equal to one, and concentrates around the origin as \( \varepsilon \) tends to zero. Since \( f \) is continuous by assumption, we have \( I_\varepsilon(t) \to f(t) \) as \( \varepsilon \downarrow 0 \). This proves the claim. \( \square \)

Remark: The additional assumption \( f, \hat{f} \in C^0(\mathbb{R}) \) is justified (by the exercise from Lecture 1): \( f \in L^1(\mathbb{R}) \) implies \( \hat{f} \in C^0(\mathbb{R}) \). To see why, let \( \omega_0 \in \mathbb{R} \) be fixed. Then

\[
\hat{f}(\omega) - \hat{f}(\omega_0) = \int f(t) (e^{-i\omega t} - e^{-i\omega_0 t}) \, dt.
\]

The integrand is dominated by the summable function \( 2|f(t)| \) so that \( \hat{f}(\omega) \to \hat{f}(\omega_0) \) as \( \omega \to \omega_0 \). Since \( \omega_0 \) is arbitrary, \( \hat{f} \in C^0(\mathbb{R}) \). The same argument can be applied to \( f \) when \( \hat{f} \in L^1(\mathbb{R}) \) using (1).

The Fourier inversion formula is valid if both \( f \) and \( \hat{f} \) are in \( L^1(\mathbb{R}) \). However, many functions of interest do not satisfy this hypothesis. For example, the boxcar function \( f(t) = \mathbb{1}_{\{-1/2 \leq t \leq 1/2\}} \) has as its Fourier transform

\[
\hat{f}(\omega) = \frac{\sin(\omega/2)}{\omega/2} \notin L^1(\mathbb{R}),
\]

and thus the above theorem does not apply. From the above remark, one can equally note that the boxcar function is not continuous. We will later discuss what to do in the case that the summability hypotheses are not satisfied, at which time we will extend the inversion formula to a larger class of functions that includes the boxcar function, among others.
3 Convolution theorem

Using the inversion formula (when it applies), we can write

\[ f(t) = \frac{1}{2\pi} \int \hat{f}(\omega)e^{i\omega t} \, d\omega. \]

If \( L \) is a TIO given by \( Lf = h \ast f \), then

\[ (Lf)(t) = \frac{1}{2\pi} \int \hat{f}(\omega)L(e^{i\omega}) (t) \, d\omega = \frac{1}{2\pi} \int \hat{f}(\omega)\hat{h}(\omega)e^{i\omega t} \, d\omega, \]

where we used the linearity of \( L \) and the fact that complex exponentials are eigenfunctions of \( L \). This expression shows that a TIO acts as a multiplication in the Fourier domain. In fact, the convolution theorem below proves the following diagram, with \( F \) denoting the Fourier transform:

\[ \begin{array}{ccc}
  f & \overset{L}{\longrightarrow} & (f \ast h)(t) \\
  F & \downarrow & \uparrow F^{-1} \\
  \hat{f}(\omega) & \overset{\times \hat{h}(\omega)}{\longrightarrow} & \hat{g}(\omega)
\end{array} \]

This diagram describes how TIOs are calculated in the real world: a signal \( f \) is decomposed as a superposition of complex exponentials, then multiplication (a very fast computation) is performed against the appropriate transfer function \( \hat{h} \), and finally the decomposition is inverted to produce \( Lf \). The invention of the Fast Fourier Transform (FFT) enables us to run this algorithm very quickly in application, with discrete signals.

**Theorem 2** (convolution theorem). Let \( f, h \in L^1(\mathbb{R}) \). Then \( g = f \ast h \) is also in \( L^1(\mathbb{R}) \), and

\[ \hat{g}(\omega) = \hat{f}(\omega)\hat{h}(\omega). \]

**Proof.** A direct calculation shows that

\[ \int |g(t)| \, dt \leq \iint |f(u)||h(t-u)| \, dudt = \left( \int |f(u)| \, du \right) \left( \int |h(t)| \, dt \right) < \infty, \]

where we used Fubini’s theorem as \( f(t)h(t-u) \) is in \( L^1(\mathbb{R} \times \mathbb{R}) \). Therefore \( g \in L^1(\mathbb{R}) \) and

\[ \hat{g}(\omega) = \int \left( \int f(u)h(t-u) \, du \right) e^{-i\omega t} \, dt \]

\[ = \iint f(u)h(t-u)e^{-i\omega t} \, dudt \]

\[ = \iint e^{-i\omega(t+u)}f(u)h(t) \, dudt \]

\[ = \hat{f}(\omega)\hat{h}(\omega), \]

where we have once again applied Fubini’s theorem.

\[ \Box \]
4 Application: proving the Central Limit Theorem

The convolution theorem actually provides a nice proof of the fundamental Central Limit Theorem (CLT) in probability theory:

**Theorem 3** (Central Limit Theorem). Assume $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables, with $E X_i = 0$ and $\text{Var} X_i = 1$. Then

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \Rightarrow N(0, 1),$$

where “$\Rightarrow$” means “converges in distribution,” and $N(0, 1)$ denotes standard normal distribution.

**Sketch of proof.** We will only consider the case when $X_i$ has a density function $f$, although the theorem holds in general. The density of $X_1 + X_2 + \cdots + X_n$ is the $n$-fold convolution

$$f^{*n}(t) = (f * \cdots * f)(t).$$

Then the density of $(X_1 + X_2 + \cdots + X_n)/\sqrt{n}$ is $\sqrt{n}f^{*n}(\sqrt{n}t)$. By the convolution theorem, the Fourier transform of this last function is $[\hat{f}(\omega/\sqrt{n})]^n$. A fact we will verify below is that

$$[\hat{f}(\omega/\sqrt{n})]^n \to e^{-\omega^2/2} \quad \text{as } n \to \infty, \quad (3)$$

independent of $f$! Since $e^{-\omega^2/2}$ is the Fourier transform of the Gaussian density with mean 0 and variance 1 (see Lecture 1), we are done by the inversion formula.

**Remark:** To prove (3), we employ the Taylor expansion

$$\hat{f}(\omega/\sqrt{n}) = \hat{f}(0) + \hat{f}'(0)\frac{\omega}{\sqrt{n}} + \frac{1}{2} \hat{f}''(0)\frac{\omega^2}{n} + \cdots$$

First notice that

$$\hat{f}(0) = \int f(t) \, dt = 1$$

since $f$ is a density function. Next recall that

$$\hat{f}'(\omega) = \int (-it)f(t)e^{-i\omega t} \, dt$$

so that

$$\hat{f}'(0) = -i \int t f(t) \, dt = -i E X_i = 0.$$

Similarly,

$$\hat{f}'(0) = i^2 \int t^2 f(t) \, dt = -\text{Var} X_i = -1.$$

We now have

$$\hat{f}(\omega/\sqrt{n}) = 1 + O\left(\frac{\omega^2}{2n} + \text{higher order terms}\right),$$

giving

$$[\hat{f}(\omega/\sqrt{n})]^n = \left(1 - \frac{\omega^2}{2n} + \text{higher order terms}\right)^n \to e^{-\omega^2/2} \quad \text{as } n \to \infty.$$
5 Properties of the Fourier transform

Table 1 lists some useful properties of the Fourier transform. In particular, the scaling property tells us that time-dilation corresponds to frequency-compression. This phenomenon is related to the uncertainty principle.

<table>
<thead>
<tr>
<th>$F(t)$</th>
<th>$\hat{F}(\omega)$</th>
<th>(transform)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(f * g)(t)$</td>
<td>$\hat{f}(\omega)\hat{g}(\omega)$</td>
<td>(convolution)</td>
</tr>
<tr>
<td>$f(t)g(t)$</td>
<td>$(2\pi)^{-1}(\hat{f} * \hat{g})(\omega)$</td>
<td>(multiplication)</td>
</tr>
<tr>
<td>$f(t - \tau)$</td>
<td>$e^{-i\omega\tau} \hat{f}(\omega)$</td>
<td>(time-shift)</td>
</tr>
<tr>
<td>$e^{i\omega_0 t} f(t)$</td>
<td>$\hat{f}(\omega - \omega_0)$</td>
<td>(modulation)</td>
</tr>
<tr>
<td>$f(t/a)$</td>
<td>$</td>
<td>a</td>
</tr>
<tr>
<td>$f^{(p)}(t)$</td>
<td>$(i\omega)^p \hat{f}(\omega)$</td>
<td>(differentiation)</td>
</tr>
<tr>
<td>$(-it)^p f(t)$</td>
<td>$\hat{f}^{(p)}(\omega)$</td>
<td>(multiplication by a monomial)</td>
</tr>
</tbody>
</table>

Table 1: Fourier transform identities