Problem 1. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$ and suppose $A$ has an SVD. Assuming that $R(B) \subseteq R(A)$, characterize all solutions to

$$AX = B$$

in terms of the SVD of $A$.

Solution. Using the SVD, write $A = U_1 S V_1^T$, where $U_1 \in \mathbb{R}^{m \times r}$, $S \in \mathbb{R}^{r \times r}$ is diagonal and invertible, and $V_1 \in \mathbb{R}^{n \times (n-r)}$. Then $A^+ = V_1 S^{-1} U_1^T$. Using theorem 6.3, solutions to $AX = B$ are of the form

$$A^+ B + (I - A^+) Y = V_1 S^{-1} U_1^T B + (I - V_1 V_1^T) Y = V_1 S^{-1} U_1^T B + (V_2 V_2^T) Y$$

for $Y$ arbitrary.
Problem 2. Consider a matrix in block form

\[ M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad A \in \mathbb{R}^{n_1 \times n_1}, D \in \mathbb{R}^{n_2 \times n_2}. \]

Show that if \( M \) is invertible, then \( A \) and \( D \) are invertible. Write down \( M^{-1} \) in block form.

Solution. Suppose \( M \) is invertible, and write \( M^{-1} \) in block form as

\[ M^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}. \]

Then we compute

\[ I_{n_1 + n_2} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix} = M M^{-1} = \begin{pmatrix} AE & AF \\ CE + DG & CF + DH \end{pmatrix}. \]

Because \( AE = I_{n_1} \), we see that \( A \) is invertible and \( E = A^{-1} \). Because \( AF = 0 \), we see that \( F = A^{-1} 0 = 0 \). Thus

\[ I_{n_1 + n_2} = \begin{pmatrix} I_{n_1} & 0 \\ CA^{-1} + DG & DH \end{pmatrix}. \]

As \( DH = I_{n_2} \), we see that \( D \) is invertible and \( H = D^{-1} \). Then we see that \( CA^{-1} + DG = 0 \), so \( G = -D^{-1} CA^{-1} \). Thus

\[ M^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1} CA^{-1} & D^{-1} \end{pmatrix}. \]
**Problem 3.** Suppose $P$ and $Q$ are orthogonal projections with $P + Q = I$. Prove that $P - Q$ must be an orthogonal matrix.

**Solution.** Because $P$ and $Q$ are orthogonal projections and $P + Q = I$, we have that

$$P - Q = (P - Q)^T = 2P - I.$$ 

Thus

$$(P - Q)^T(P - Q) = (2P - I)(2P - I) = 4P^2 - 4P + I = I.$$
Problem 4. Suppose that a matrix $A \in \mathbb{R}^{m \times n}$ has linearly independent columns. Prove that the orthogonal projection onto the space spanned by these columns is given by the matrix $P = A(A^T A)^{-1} A^T$.

Solution. As in problem 1, write $A = U_1 SV_1^T$. Then the projection onto the column space is given by $U_1 U_1^T$. Now we compute

$$A(A^T A)^{-1} A^T = U_1 S V_1^T (V_1 S^2 V_1^T)^{-1} V_1 S U_1^T = U_1 U_1^T.$$ 

We could also note that the orthogonal projection onto $R(A)$ is $AA^+$. By example 4.6, we have that $A^+ = (A^T A)^{-1} A^T$. Thus $AA^+ = A(A^T A)^{-1} A^T$.

Alternatively, we first check that $P$ is an orthogonal projection. We compute

$$P^2 = [A(A^T A)^{-1} A^T][A(A^T A)^{-1} A^T] = A(A^T A)^{-1} [(A^T A)(A^T A)^{-1}] A^T = A(A^T A)^{-1} A^T = P,$$

and

$$P^T = [A(A^T A)^{-1} A^T]^T = A^T (A^T A)^{-T} A^T = A(A^T A)^{-1} A^T = P.$$ 

Now it suffices to show that the range of $P$ is the range of $A$. If $P x = y$, then $A[(A^T A)^{-1} A^T x] = y$, so $R(P) \subseteq R(A)$. It suffices to show that the rank of $P$ is equal to the rank of $A$. This follows from the fact that the rank of $A$ is equal to the rank of $A^T$ and that $A$ and $(A^T A)^{-1}$ are one-to-one.
Problem 5.  (a) Let $D$ be a $10,000 \times 10,000$ diagonal matrix and let $F$ be a $10,000 \times 10$ matrix.

- Show that
  \[(D + FF^T)^{-1} = D^{-1} - D^{-1}F(I_{10} + F^T D^{-1}F)^{-1}F^T D^{-1}.\]

- Suppose I want to compute the inverse of $D + FF^T$. Why is this formula interesting?

(b) Programming question.

Solution.  (a) Recall the Sherman-Morrison-Woodbury formula, which states that if $G \in \mathbb{R}^{n \times n}$ is invertible, $B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}$, and $E \in \mathbb{R}^{m \times m}$ are matrices such that $E^{-1} + CG^{-1}B$ is invertible, then

\[(G + BCE)^{-1} = G^{-1} - G^{-1}B(E^{-1} + CG^{-1}B)^{-1}CG^{-1}.\]

Using the special case $G = D, B = F, E = I_{10}$ and $C = F^T$ gives

\[(D + FF^T)^{-1} = D^{-1} - D^{-1}F(I_{10} + F^T D^{-1}F)^{-1}F^T D^{-1}.\]

This is useful because it reduces inverting a $10,000 \times 10,000$ matrix into inverting a diagonal matrix (which is easy) and a $10 \times 10$ matrix, $I_{10} + F^T D^{-1}F$.

(b) The programming part will be explicitly treated in class so please stay tuned.