Stat/219 Math 136 - Stochastic Processes

Notes on Markov Processes

1 Notes on Markov processes

The following notes expand on Proposition 6.1.17.

1.1 Stochastic processes with independent increments

Lemma 1.1 *If $\{X_t, t \geq 0\}$ is a stochastic process with independent increments then $\{X_t, t \geq 0\}$ is a Markov process.*

Proof: Let $\{\mathcal{F}_t, t \geq 0\}$ be the canonical filtration of $\{X_t, t \geq 0\}$. Independent increments means that for any $t, h \geq 0$, the random variable $X_{t+h} - X_t$ is indenpendent of $\mathcal{F}_t$. To show that $\{X_t\}$ is a Markov process, from Definition 6.1.10 it suffices to show that

$$\mathbb{E}[f(X_{t+h})|\mathcal{F}_t] = \mathbb{E}[f(X_{t+h})|X_t]$$

for any bounded measurable function f on $(\mathbb{S}, \mathcal{B})$ (where $\mathbb{S}$ is the state space and $\mathcal{B}$ is its Borel σ-field). Let f be such a function. Then

$$\mathbb{E}[f(X_{t+h})|\mathcal{F}_t] = \mathbb{E}[f(X_{t+h} - X_t + X_t)|\mathcal{F}_t] = \mathbb{E}[f(X_{t+h} - X_t + X_t)|X_t],$$

where the last equality follows upon noting that X_t is $\mathcal{F}_t$-measurable and $X_{t+h} - X_t$ is indenpendent of $\mathcal{F}_t$. ■

The converse of Lemma 1.1 is not true. Try to find a counterexample. (Hint: consider a process from Exercise 6.1.19.)

1.2 Homogeneous Markov processes

The main result is Proposition 6.1.17: *If $\{X_t, t \geq 0\}$ has stationary and independent increments then $\{X_t, t \geq 0\}$ is a homogeneous Markov process.*

The idea of the proof is similar to the proof of Lemma 1.1 above.

One further result is:

Lemma 1.2 *If $\{X_t, t \geq 0\}$ is a stationary process and a Markov process then it is homogeneous.*

Proof: We only show here the case of a discrete time, countable state process $\{X_n\}$. Note that by stationarity, X_n has the same distribution as X_0 and (X_{n+1}, X_n) has the same distribution as (X_1, X_0). Thus

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \frac{\mathbb{P}(X_{n+1} = y, X_n = x)}{\mathbb{P}(X_n = x)} = \frac{\mathbb{P}(X_1 = y, X_0 = x)}{\mathbb{P}(X_0 = x)} = \mathbb{P}(X_1 = y | X_0 = x),$$

which shows that the transition probabilities do not depend on n, i.e. the chain is homogeneous. ■

Note that in Lemma 1.2 we assumed that process itself is stationary, which is a stronger assumption than just having stationary increments. The following example illustrates why stationary increments is not enough.

If a Markov process has stationary increments, it is not necessarily homogeneous. Consider the Brownian bridge $B_t = W_t - tW_1$ for $t \in [0,1]$. In Exercise 6.1.19 you showed that $\{B_t\}$ is a Markov process which is not homogeneous. We now show that it has stationary increments. Since $\{B_t\}$ is a Gaussian process (see Exercise 5.1.7) the random variable $B_{t+h} - B_t = W_{t+h} - W_t - hW_1$ is Gaussian (for $0 \le t \le t+h \le 1$). It suffices to show that its mean and variance do not depend on t. Clearly it has mean 0. Also,

$$\mathbb{E}|B_{t+h} - B_t|^2 = \mathbb{E}|W_{t+h} - W_t|^2 + h^2\mathbb{E}|W_1|^2 - 2h\mathbb{E}[(W_{t+h} - W_t)W_1] = h + h^2 - 2h(h) = h - h^2.$$

Note that $\{B_t\}$ does not have independent increments (which you should check).

The next example shows that the converse to Lemma 1.2 is not true. The example actually shows that homogeneity does not imply stationary increments from which it follows that the converse to Lemma 1.2 is not true (since any stationary process must have stationary increments.)

If a Markov process is homogeneous, it does not necessarily have stationary increments. Consider state space $\mathbb{S} = \{0, 1, 2, \ldots\}$, initial distribution π with $\pi(\{0\}) = 1$, and stationary transition probability function: $p(\{1\}|0) = 1$ and for $x \in \mathbb{S} \setminus \{0\}$,

$$p(A|x) = \begin{cases} \frac{1}{x+1}, & A = \{x+1\}, \\ \frac{x}{x+1}, & A = \{x-1\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then there exists on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a discrete homogeneous Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\mathbb{S}$, intial distribution π (i.e. $\mathbb{P}(X_0 = 0) = 1$) and transition function $p(A|x)$, i.e. $\mathbb{P}(X_{n+1} \in A | \mathcal{F}_n) = p(A|X_n)$ a.s., where $\{\mathcal{F}_n\}$ is the canonical filtration of $\{X_n\}$. Note that since $\mathbb{P}(X_0 = 0) = 1$ we have

$$\mathbb{P}(X_1 = 1) = \sum_{x \in \mathbb{S}} \mathbb{P}(X_1 = 1 | X_0 = x)\mathbb{P}(X_0 = x) = \sum_{x \in \mathbb{S}} p(\{1\}|x)\mathbb{P}(X_0 = x) = p(\{1\}|0) = 1.$$

It follows that $\mathbb{P}(X_1 - X_0 = 1) = 1$. Also, again since $\mathbb{P}(X_1 = 1) = 1$, we have

$$\mathbb{P}(X_2 - X_1 = 1) = \sum_{x \in \mathbb{S}} \mathbb{P}(X_2 - X_1 = 1 | X_1 = x)\mathbb{P}(X_1 = x) = \sum_{x \in \mathbb{S}} p(\{x+1\}|x)\mathbb{P}(X_1 = x) = p(\{2\}|1) = 1/2.$$

Thus $X_1 - X_0$ does not have the same distribution as $X_2 - X_1$ and so $\{X_n\}$ does not have stationary increments.

Intuitively, if a Markov process $\{X_t\}$ is homogeneous, then the conditional distribution of $X_{t+h} - X_t$ given X_t does not depend on t. Conditional on X_t, X_t is treated like a known constant so all the randomness is given by the change from the known value X_t to the uncertain value X_{t+h}. The same is not necessarily true for the unconditional distribution of $X_{t+h} - X_t$ in which X_t itself is random and has a distribution which might depend on t (even if $\{X_t\}$ has stationary increments).

1.3 Showing that a stochastic process is a Markov process

We have seen three main ways to show that a process $\{X_t, t \geq 0\}$ is a Markov process:

1. Compute $\mathbb{P}(X_{t+h} \in A | \mathcal{F}_t)$ directly and check that it only depends on X_t (and not on $X_u, u < t$).
2. Show that the process has independent increments and use Lemma 1.1 above.
3. Show that it is a function of another Markov process and use results from lecture about functions of Markov processes (e.g. if f is invertible and $\{Y_t\}$ is a Markov process then $\{f(Y_t)\}$ is a Markov process).