Statistics 116 - Fall 2004
Theory of Probability
Midterm # 2, Practice # 2
SHOW (AND BRIEFLY EXPLAIN) ALL OF
YOUR WORK.
CALCULATORS ARE PERMITTED FOR
NUMERICAL CALCULATIONS ONLY.

Instructions: Answer 4 out of 5 questions. Clearly mark which 4 questions you decide to answer. If you do not clearly indicate which 4 are to be counted, your mark will be based on 5 instead of 4 questions, there are no bonus points. All questions have equal weight.

Q. 1) The density function of $X$ is given by

$$f(x) = \begin{cases} 
  a + bx^2 & 0 \leq x \leq 1 \\
  0 & \text{otherwise.}
\end{cases}$$

If $E(X) = 3/5$, find $a$ and $b$.

A: We know that

$$1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{1} (a + bx^2) \, dx = a + \frac{b}{3}.$$ 

We are also given

$$\frac{3}{5} = \int_{0}^{1} x(a + bx^2) \, dx = \frac{a}{2} + \frac{b}{4}.$$ 

We therefore have to solve the system of equations

$$3a + b = 3$$

$$10a + 5b = 12$$

for $a$ and $b$. The solution is given by $a = 3/5$, $b = 6/5$. 

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Q. 2) If $X$ is a Poisson random variable with parameter $\lambda$, show that

$$E(X^n) = \lambda E((X + 1)^{n-1}).$$

Use this result to compute $E(X^3)$.

A: We know that

$$E(X^n) = e^{-\lambda} \sum_{j=0}^{\infty} \frac{j^n \lambda^j}{j!}$$

$$= e^{-\lambda} \sum_{j=1}^{\infty} \frac{j^n \lambda^j}{j!}$$

$$= e^{-\lambda} \sum_{j=1}^{\infty} \frac{j^{n-1} \lambda^j}{(j-1)!}$$

$$= \lambda e^{-\lambda} \sum_{j=1}^{\infty} \frac{j^{n-1} \lambda^j}{(j-1)!}$$

$$= \lambda \left( e^{-\lambda} \sum_{i=0}^{\infty} (i+1)^{n-1} \frac{\lambda^i}{i!} \right)$$

$$= \lambda \cdot E((X + 1)^{n-1}).$$

Because $\text{Var}(X) = \lambda$, we know that $E(X^2) = \lambda^2 + \lambda$. Therefore,

$$E(X^3) = \lambda \cdot E((X + 1)^2)$$

$$= \lambda \cdot E(X^2 + 2X + 1) = \lambda \cdot (\lambda^2 + \lambda + 2 \cdot \lambda + 1)$$

$$= \lambda^3 + 3\lambda^2 + \lambda.$$
Q. 3) The lifetime of a certain battery has p.d.f.

\[ f(x) = \begin{cases} 
3x^2e^{-x^3} & x \geq 0 \\
0 & x < 0.
\end{cases} \]

(a) Find the hazard rate function of \( X \).

(b) Use part (a), or another method, to compute the probability that the battery lasts 1.5 hours given that it has lasted 1 hour.

**A:**

(a) The hazard is given by

\[ h(x) = \frac{f(x)}{1 - F(x)} \]

but

\[ 1 - F(x) = \int_x^\infty 3t^2e^{-t^3} \, dt \]

\[ = -e^{-t^3}\bigg|_x^\infty \]

\[ = e^{-x^3} \]

Therefore

\[ h(x) = 3x^2. \]

(b)

\[ P(X > 1.5 | X > 1.0) = e^{-\int_{1.0}^{1.5} 3x^2 \, dx} = e^{-((1.5^3) - 1)}. \]
Q. 4) Suppose your ISP charges a flat rate of $19.95 per month for 50 hours and an additional $1 per hour afterwards. If your internet use is approximately an exponential random variable with parameter 1/40. What is your average monthly internet bill?

A: If $U$ represents your internet usage, then

\[
B(U) = \begin{cases} 
19.95 & U < 50 \\
19.95 + (U - 50) & U \geq 50
\end{cases}
\]

represents your monthly bill. Therefore

\[
E(B(U)) = \frac{1}{40} \left( \int_0^{50} 19.95e^{-t/40} \, dt + \int_{50}^{\infty} (19.95 + t - 50)e^{-t/40} \, dt \right)
\]

\[
= 19.95 + \frac{1}{40} \int_{50}^{\infty} (t - 50)e^{-t/40} \, dt
\]

\[
= 19.95 + \frac{1}{40} \int_{50}^{\infty} te^{-t/40} \, dt - 50e^{-50/40}
\]

\[
= 19.95 + \left( -te^{-t/40} \bigg|_{50}^{\infty} + \int_{50}^{\infty} e^{-t/40} \, dt \right) - 50e^{-50/40}
\]

\[
= 19.95 + \left( 50e^{-50/40} - 40e^{-t/40} \bigg|_{50}^{\infty} \right) - 50e^{-50/40}
\]

\[
= 19.95 + 40e^{-50/40}.
\]
Q. 5) Let
\[ f(x, y) = 24xy, \quad 0 \leq x \leq 0, 0 \leq y \leq 1, 0 \leq x + y \leq 1 \]
and let it equal 0 otherwise.

(a) Show that \( f(x, y) \) is a joint probability density function.
(b) Find \( E(X) \).
(c) Find \( E(Y) \).

A:

(a) To show that it is density function, we must show that
\[ \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1, f(x, y) \geq 0. \]

Clearly, \( f(x, y) \geq 0 \) so we turn to the integral.
\[
\int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1-x} 24xy \, dy \, dx \\
= \int_{0}^{1} 12x(1-x)^2 \, dx \\
= \int_{0}^{1} 12x^3 - 24x^2 + 12x \, dx \\
= 3 - 8 + 6 \\
= 1.
\]

(b) To compute \( E(X) \), we first compute the marginal density of \( X \)
\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy \\
= \int_{0}^{1-x} 24xy; \, dy \\
= 12x(1-x)^2.
\]

Therefore,
\[
E(X) = \int_{0}^{1} x \cdot 12x(1-x)^2 \, dx \\
= \int_{0}^{1} 12x^4 - 24x^3 + 12x^2 \, dx \\
= \frac{12}{5} - 6 + 4 \\
= \frac{2}{5}.
\]
Alternatively, we note that the marginal density of $X$ is Beta(2, 3) so the expected value of $X$ is

$$E(X) = \frac{a}{a+b} = \frac{2}{2+3} = \frac{2}{5}.$$ 

(c) By symmetry, the marginal density of $Y$ is the same as that of $X$. Therefore, $E(Y) = 2/5$. 