Statistics 116 - Fall 2004
Theory of Probability
Midterm # 2, Practice # 1
SHOW (AND BRIEFLY EXPLAIN) ALL OF
YOUR WORK.
CALCULATORS ARE PERMITTED FOR
NUMERICAL CALCULATIONS ONLY.

Instructions: Answer 4 out of 5 questions. Clearly mark which 4 questions
you decide to answer. If you do not clearly indicate which 4 are to be counted,
your mark will be based on 5 instead of 4 questions, there are no bonus points.
All questions have equal weight.

Q. 1) Let $X$ be a Binomial random variable with parameters $n$ and $p$. Show
that

$$E \left( \frac{1}{X + 1} \right) = \frac{1 - (1 - p)^{n+1}}{(n + 1)p}.$$  

Solution:

$$E \left( \frac{1}{X + 1} \right) = \sum_{j=0}^{n} \frac{1}{j + 1} \binom{n}{j} p^j (1 - p)^{n-j}$$

$$= \sum_{j=0}^{n} \frac{n!}{(n-j)!(j+1)!} p^j (1 - p)^{n-j}$$

$$= \frac{1}{(n + 1)p} \sum_{j=0}^{n} \frac{(n + 1)!}{(n-j)!(j+1)!} p^{j+1} (1 - p)^{n-j}$$

$$= \frac{1}{(n + 1)p} \sum_{j=1}^{n+1} \frac{(n + 1)!}{(n + 1 - j)!(j)!} p^j (1 - p)^{n+1-j}$$

$$= \frac{1}{(n + 1)p} \left( 1 - (1 - p)^{n+1} \right)$$
Q. 2) A filling station is supplied with gasoline once a week. Suppose its weekly volume of sales in thousands of gallons is a random variable with probability density function

\[ f(x) = \begin{cases} 
5(1-x)^4 & 0 < x < 1 \\
0 & \text{otherwise.}
\end{cases} \]

How large must the capacity of the tank be so that the probability of the supply’s being exhausted in a given week is 0.01?

Solution:

If the tank’s size is \( c \), the probability it will be exhausted is

\[ P(X \geq c) = \int_c^1 5(1-x)^4 \, dx \]

\[ = (1-c)^5. \]

For this to be 0.01, we see that

\[ (1-c)^5 = 0.01 \iff c = 1 - 0.01^5 = 0.60. \]
Q. 3) Suppose that the life distribution of an item has hazard rate function

\[ \lambda(t) = \frac{(t - 1)^4}{2} + 1, \quad t > 0. \]

What is the probability that

(a) the item survives to age 2;
(b) the item’s lifetime is between 0.4 and 1.4;
(c) a 1 year-old item will survive to age 2?

Solution:

(a) 
\[
P(X > 2) = 1 - F(2) = e^{- \int_2^0 \lambda(t) \, dt} = e^{- \left[ \frac{(t-1)^5}{10} + t \right]_0^2} = e^{-2/10 - 2} = e^{-2.2} = 0.111
\]

(b) 
\[
P(0.4 < X \leq 1.4) = F(1.4) - F(0.4) = e^{- \int_0^{1.4} \lambda(t) \, dt} - e^{- \int_0^{0.4} \lambda(t) \, dt} = e^{-0.99224} - e^{-1.50102} = 0.388
\]

(c) 
\[
P(X > 2 | X > 1) = e^{- \int_1^2 \lambda(t) \, dt} = e^{-1.1} = 0.332
\]
Q. 4) A telephone company has the following calling plan: for a flat rate of 39.95$ a month, you receive 1000 minutes a month of prepaid cellular time and all additional minutes will be billed at 0.10$ a minute. If the number of minutes you use is approximately normally distributed with mean 900 and variance 10000. What is your average monthly phone bill?

**Solution:** Let $B(X)$ be your monthly bill:

$$B(X) = \begin{cases} 
39.95 & X \leq 1000 \\
39.95 + (X - 1000) \cdot 0.1 & X > 1000 
\end{cases}$$

Then,

$$E(B(X)) = \int_{-\infty}^{1000} 39.95 \frac{e^{-(x-900)^2/(2 \cdot 10000)}}{100 \cdot \sqrt{2\pi}} \, dx +$$

$$\int_{1000}^{\infty} (39.95 + 0.1 \cdot (x - 1000)) \frac{e^{-(x-900)^2/(2 \cdot 10000)}}{100 \cdot \sqrt{2\pi}} \, dx$$

$$= 39.95 + \int_{1000}^{\infty} 0.1 \cdot (x - 1000) \frac{e^{-(x-900)^2/(2 \cdot 10000)}}{100 \cdot \sqrt{2\pi}} \, dx$$

$$= 39.95 + \int_{1}^{\infty} 10 \cdot (u - 1) \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du$$

$$= 39.95 + 10 \cdot \left( \frac{e^{-1/2}}{\sqrt{2\pi}} - (1 - \Phi(1)) \right)$$

$$= 39.95 + 10 \cdot \left( \frac{e^{-1/2}}{\sqrt{2\pi}} - (1 - \Phi(1)) \right)$$

$$= 40.78$$
Q. 5) The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = e^{-(x+y)}, \quad 0 \leq x < \infty, 0 \leq y < \infty.$$ 

(a) Find $P(X < Y)$

(b) Find $P(X < a)$.

Solution:

(a)

$$P(X < Y) = \int_{-\infty}^{\infty} \int_{x}^{\infty} f(x, y) \, dy \, dx$$

$$= \int_{0}^{\infty} \int_{x}^{\infty} e^{-(x+y)} \, dy \, dx$$

$$= \int_{0}^{\infty} e^{-x} \cdot \left( \int_{x}^{\infty} e^{-y} \, dy \right) \, dx$$

$$= \int_{0}^{\infty} e^{-x} \cdot e^{-x} \, dx$$

$$= \int_{0}^{\infty} e^{-2x} \, dx$$

$$= \frac{1}{2}.$$

Note: This shouldn’t be surprising: above, $X$ and $Y$ are independent and have the same marginal distribution: Exp(1). Because they have the same distribution there should be a 50-50 chance that $X$ is the smaller of the two. The same reasoning says if you have $(X_1, \ldots, X_n)$ all independent with the same distribution, then the probability that $X_j$ is smaller than all the other $X$’s is $1/n$. This generalizes to dependent random variables $(X_1, \ldots, X_n)$ whose distribution is symmetric, i.e. for every permutation $\pi$ the distribution of $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ is the same as $(X_1, \ldots, X_n)$. Such random vectors are called exchangeable.

(b)

$$P(X < a) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx$$

$$= \int_{-\infty}^{a} \left( \int_{-\infty}^{\infty} f(x, y) \, dy \right) \, dx$$

$$= \int_{-\infty}^{a} f_X(x) \, dx$$

$$= \int_{0}^{a} \int_{0}^{\infty} e^{-(x+y)} \, dy \, dx$$

$$= \int_{0}^{a} e^{-x} \cdot \left( \int_{0}^{\infty} e^{-y} \, dy \right) \, dx$$

$$= 1 - e^{-a}.$$