**Statistics 116 - Fall 2004**

**Theory of Probability**

**Final Exam, December 10th, 2004**

**Solutions**

**Instructions:** Answer Q. 1-6. All questions have equal weight. The exam is open book. In addition, you are allowed a maximum of 3 pages of handwritten notes.

Q. 1) Let \( X \) be a continuous random variable with density function

\[
f_X(x) = \frac{4}{3} \left( \frac{x}{3} \right)^3 \exp \left( -\left( \frac{x}{3} \right)^4 \right), \quad x \geq 0
\]

for constants \( \alpha, \beta > 0 \).

(a) Compute \( h_X(x) \), the hazard rate function of \( X \).

(b) Compute \( P(X > 4 | X > 3) \).

**Solution:**

(a) By definition

\[
h_X(x) = \frac{f_X(x)}{1 - F_X(x)}
\]

and

\[
1 - F_X(x) = \int_x^\infty f_X(t) \, dt
\]

\[
= \int_x^\infty \frac{4}{3} \left( \frac{t}{3} \right)^3 \exp \left( -\left( \frac{t}{3} \right)^4 \right) \, dt
\]

\[
= - \exp \left( -\left( \frac{t}{3} \right)^4 \right) \bigg|_x^\infty
\]

\[
= \exp \left( -\left( \frac{x}{3} \right)^4 \right)
\]

Therefore,

\[
h_X(x) = \frac{\frac{4}{3} \left( \frac{x}{3} \right)^3 \exp \left( -\left( \frac{x}{3} \right)^4 \right)}{\exp \left( -\left( \frac{x}{3} \right)^4 \right)} = \frac{4}{3} \left( \frac{x}{3} \right)^3.
\]
Q. 2) Let $X \sim \text{Geometric}(p)$ be a Geometric random variable.

(a) Show that $P(X > n) = (1 - p)^n$.

HINT: USE THE FACT THAT $\sum_{j=0}^{\infty} p(1 - p)^{j-1} = 1$.

(b) Show that $X$ has the following memoryless property:
   For any integers $n$ and $m$, with $n > m$
   
   $$P(X > n | X > m) = P(X > n - m).$$

Solution:

(a) 

$$P(X > n) = P(\text{first } n \text{ trials were failures}) = (1 - p)^n.$$

(b) For $n > m$

$$P(X > n | X > m) = \frac{P(X > n, X > m)}{P(X > m)}$$

$$= \frac{P(X > n)}{P(X > m)}$$

$$= \frac{(1 - p)^n}{(1 - p)^m}$$

$$= (1 - p)^{n-m}$$

$$= P(X > n - m).$$
Q. 3) A fair die is rolled \( n \) times with the results denoted by \((R_i)_{i \geq 1}\).

(a) For \( n \) fixed, show that the expected value of the sum of the first \( n \) rolls

\[ E \left( \sum_{i=1}^{n} R_i \right) = n \frac{7}{2} \]

(b) For \( n \) fixed, show that the variance of the sum of the first \( n \) rolls

\[ E \left( \sum_{i=1}^{n} R_i \right) = n \frac{35}{12} \]

(c) Suppose that instead of fixing \( n \), the die is continually rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are needed.

(a)

\[ E(\sum_{i=1}^{n} R_i) = \sum_{i=1}^{n} E(R_i) = \sum_{i=1}^{n} \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \sum_{i=1}^{n} \frac{7}{2} = n \frac{7}{2}. \]

(b)

By independence,

\[ \text{Var}(\sum_{i=1}^{n} R_i) = \sum_{i=1}^{n} \text{Var}(R_i) = n \text{Var}(R_1). \]

\[ \text{Var}(R_1) = \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) - \frac{49}{4} = \frac{35}{12} \]

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Q. 4) The joint density function of $X$ and $Y$ is given by

$$f(x, y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}} \cdot ye^{-y}, \quad -\infty < x < \infty, y > 0.$$  

(a) Find the conditional density $f_{X|Y}(x|y)$ of $X$ given $Y = y$.
(b) Compute $E(X|Y)$.
(c) Compute $\text{Var}(X|Y)$.
(d) Compute $\text{Var}(X)$.

**Solution:**

(a) By inspection, it is not hard to see that

$$f_Y(y) = ye^{-y}, \quad y \geq 0$$

and

$$f_{X|Y}(x|y) = \frac{e^{-yx^2/2}}{\sqrt{2\pi/y}}$$

is a Normal density with mean 0 and variance $1/y$. Or,

$$X|Y = y \sim N(0, 1/y).$$

(b) As the conditional distribution is $N(0, 1/Y)$

$$\text{Var}(X|Y) = 1/Y.$$  

(c) As the conditional distribution is $N(0, 1/Y)$

$$E(X|Y) = 0.$$  

(d) Similarly,

$$\text{Var}(X|Y) = 1/Y.$$  

(e)

$$\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$$

$$= E(1/Y) + \text{Var}(0)$$

$$= \int_0^\infty \frac{1}{y} ye^{-y} \, dy$$

$$= \int_0^\infty e^{-y} \, dy$$

$$= \int_0^\infty e^{-y} \, dy$$

$$= 1.$$
Q. 5) Suppose that $X$ and $Y$ are independent $\text{Exp}(\lambda)$ random variables and let 

$$Z = \frac{X}{X+Y}.$$ 

Show that, for $0 < z < 1$ 

$$F_Z(z) = P(Z \leq z) = z,$$

i.e. the random variable $Z$ is uniformly distributed over $(0, 1)$.

**Hint:** express $F_Z(z)$ as a double integral.

**Solution:**

For $0 < z < 1$

\[
P(Z \leq z) = P(X/(X + Y) \leq z) = P(X \leq Xz + Yz) = P(X(1 - z)/z \leq Y) = P((X,Y) \in \{(x,y) : x(1 - z)/z \leq y\}) = \int_{0}^{\infty} \left( \int_{x(1-z)/z}^{\infty} \lambda e^{-\lambda y} \, dy \right) \lambda e^{-\lambda x} \, dx
\]

\[
= \int_{0}^{\infty} e^{-\lambda x(1-z)/z} \lambda e^{-\lambda x} \, dx = \int_{0}^{\infty} \lambda e^{-\lambda x/z} \, dx = z.
\]
Q. 6) A coin having probability $p$ of coming up heads is continually flipped until both heads and tails have appeared. Let $X$ denote the total number of flips necessary.

(a) What is the probability that the last flip lands heads?

(b) Argue that conditional on the first flip being a head, $X - 1$ has a $\text{Geometric}(1 - p)$ distribution. That is, if

$$H = \{\text{first flip results in heads}\}$$

then

$$X - 1 | H \sim \text{Geometric}(1 - p).$$

(c) Argue that conditional on the first flip being a tail

$$X - 1 | T \sim \text{Geometric}(p),$$

with

$$T = \{\text{first flip results in tails}\}.$$

(d) Show that

$$E(X) = E(X | H) P(H) + E(X | H^c) P(H^c)$$

$$= E(X | H) P(H) + E(X | T) P(T)$$

$$= \left(1 + \frac{1}{1 - p}\right) p + \left(1 + \frac{1}{p}\right) (1 - p)$$

$$= 1 + \frac{p}{1 - p} + \frac{1 - p}{p}. $$

Solution:

(a) Let

$$E = \{\text{last flip is heads}\}.$$

Then

$$E = \{\text{first flip is tails}\}.$$

Therefore,

$$P(E) = 1 - p.$$ 

(b) If the first flip is heads, then 1 flip has been completed and what remains to occur is for a tail to occur. Therefore, the remaining amount, minus the first 1 flip, is Geometric with parameter $1 - p$.

(c) Similar to (b).

(d) Show that

$$E(X) = E(X | H) P(H) + E(X | H^c) P(H^c)$$

$$= E(X | H) P(H) + E(X | T) P(T)$$

$$= \left(1 + \frac{1}{1 - p}\right) p + \left(1 + \frac{1}{p}\right) (1 - p)$$

$$= 1 + \frac{p}{1 - p} + \frac{1 - p}{p}. $$