Statistics 116 - Fall 2004
Theory of Probability
Alternate Final Exam, December 7th, 2004
Solutions

Instructions: Answer Q. 1-6. All questions have equal weight. Bonus worth equivalent of one half question. The exam is open book. In addition, you are allowed a maximum of 3 pages of handwritten notes.

Q. 1) Let $X \sim \text{Geom}(p)$ be a Geometric random variable. Show that $X$ has the following memoryless property:
For any integers $n$ and $m$, with $m < n$

$$P(X > n | X > m) = P(X > n - m).$$

Solution:

$$P(X > n) = P(\text{first } n \text{ trials were failures}) = (1 - p)^n.$$ Therefore, for $n > m$

$$P(X > n | X > m) = \frac{P(X > n, X > m)}{P(X > m)}$$

$$= \frac{P(X > n)}{P(X > m)}$$

$$= \frac{(1 - p)^n}{(1 - p)^m}$$

$$= (1 - p)^{n-m}$$

$$= P(X > n - m).$$
Q. 2) The joint density function of \(X\) and \(Y\) is given by
\[
f(x, y) = \frac{e^{-(x-y)^2/2y}}{\sqrt{2\pi y}} \frac{y^2e^{-y}}{2}, \quad -\infty < x < \infty, y \geq 0.
\]
(a) Find the conditional density \(f_{X|Y}(x|y)\) of \(X\) given \(Y = x\).
(b) Compute Var\((X|Y)\).
(c) Compute \(E(X|Y)\).
(d) Compute Var\((X)\).

Solution:

(a) By inspection, it is not hard to see that
\[
f_Y(y) = \frac{y^2e^{-y}}{2}, \quad y \geq 0
\]
and
\[
f_{X|Y}(x|y) = \frac{e^{-(x-y)^2/2y}}{\sqrt{2\pi y}}
\]
is a Normal density with mean \(y\) and variance \(y\). Or,
\[
X|Y = y \sim N(y, y).
\]
(b) As the conditional distribution is \(N(Y, Y)\)
\[
\text{Var}(X|Y) = Y.
\]
(c) Similarly,
\[
E(X|Y) = Y.
\]
(d) \[
\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))
\]
\[
= E(Y) + \text{Var}(Y)
\]
\[
= E(Y) + E(Y^2) - E(Y)^2
\]
\[
E(Y) = \int_0^\infty \frac{y^3e^{-y}}{2} dy
\]
\[
= \frac{\Gamma(4)}{\Gamma(3)}
\]
\[
= 3
\]
\[
E(Y^2) = \int_0^\infty \frac{y^4e^{-y}}{2} dy
\]
\[
= \frac{\Gamma(5)}{\Gamma(3)}
\]
\[
= 12
\]
Therefore,
\[
\text{Var}(X) = 3 + 12 - 3^2 = 6.
\]
Q. 3) Suppose that $X$ is a continuous random variable with density
\[ f_X(x) = \alpha \lambda e^{-\lambda x} + (1 - \alpha) \mu e^{-\mu x}, \quad x \geq 0 \]
where $0 < \alpha < 1$ and $0 < \lambda < \mu$ (in this case the distribution of $X$ is said to be a mixture of an Exp($\lambda$) and an Exp($\mu$) distribution).

(a) Compute $h_X(t)$, the hazard rate of $X$.
(b) Recalling that we have assumed $\lambda < \mu$, show that
\[ \lim_{t \to \infty} h_X(t) = \lambda. \]

Solution:
(a)
\[
h_X(t) = \frac{f_X(t)}{1 - F_X(t)}
\]
\[
1 - F_X(t) = \int_t^\infty f_X(s) \, ds
= \alpha \int_t^\infty \lambda e^{-\lambda s} \, ds + (1 - \alpha) \int_t^\infty \mu e^{-\mu s} \, ds
= \alpha e^{-\lambda t} + (1 - \alpha) e^{-\mu t}
\]
\[
h_X(t) = \frac{\alpha \lambda e^{-\lambda t} + (1 - \alpha) \mu e^{-\mu t}}{\alpha e^{-\lambda t} + (1 - \alpha) e^{-\mu t}}
\]
(b)
\[
\lim_{t \to \infty} h_X(t) = \lim_{t \to \infty} \frac{\alpha \lambda e^{-\lambda t} + (1 - \alpha) \mu e^{-\mu t}}{\alpha e^{-\lambda t} + (1 - \alpha) e^{-\mu t}}
= \lim_{t \to \infty} \frac{\alpha \lambda + (1 - \alpha) \mu e^{(\lambda - \mu) t}}{\alpha + (1 - \alpha) e^{(\lambda - \mu) t}}
= \lambda
\]
because for $\lambda < \mu$
\[ \lim_{t \to \infty} e^{(\lambda - \mu) t} = 0. \]
Q. 4) Given a sequence \((X_1, \ldots, X_n)\) of \(n\) independent, identically distributed random variables we say a record occurred at time \(i\) if \(X_i \geq X_j, 1 \leq j \leq i - 1\). That is, there is a record at time \(i\), if, at time \(i\), \(X_i\) is the “record” largest of the first \(i\) elements of the sequence.

(a) For \(1 \leq i \leq n\), compute \(p_i = P(X_i \text{ is a record})\)

(b) Let \(R_n\) be the total number of records of the sequence \((X_1, \ldots, X_n)\).

Compute \(E(R_n)\).

**Solution:**

(a)

\[
P(X_i \text{ is a record}) = P(X_i \geq X_1, X_i \geq X_2, \ldots, X_i \geq X_{i-1}).
\]

This just says that if we order the \(X\)’s from 1 to \(i\) then \(X_i\) is the last one. As there are \(i!\) such orderings and \((i - 1)!\) orderings with \(X_i\) the last entry,

\[
P(X_i \text{ is a record}) = \frac{1}{i}.
\]

(b) Let

\[
Y_i = \begin{cases} 
1 & \text{if } X_i \text{ is a record} \\
0 & \text{otherwise.}
\end{cases}
\]

Then,

\[
R_n = \sum_{i=1}^{n} Y_i
\]

and

\[
E(R_n) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} P(X_i \text{ is a record}) = \sum_{i=1}^{n} \frac{1}{i}.
\]
Q. 5) A model for the movement of a stock supposes that if the present price of the stock is \( s \), then after one time period it will be either \( u \times s \) with probability \( p \) or \( d \times s \) with probability \( 1 - p \). Assuming that successive movements are independent, approximate the probability that the stock’s price will be up at least 30 percent after the next 1000 time periods if \( u = 1.012, d = 0.990 \) and \( p = 0.51 \).

(HINT: \( \log(ab) = \log(a) + \log(b) \)).

Solution:
Define the random variables

\[
X_i = \begin{cases} 
1 & \text{if stock rises at } i\text{-th time period} \\
0 & \text{otherwise.} 
\end{cases}
\]

Then, the \( X_i \)’s are independent with

\[ P(X_i = 1) = p \]

and the price of the stock \( S_n \) at the \( n \)-th time period is

\[
S_n = S_0 u^{\sum_{i=1}^{n} X_i} d^{n - \sum_{i=1}^{n} X_i} = S_0 d^n \left( \frac{u}{d} \right)^{\sum_{i=1}^{n} X_i}.
\]

The event that the stock has risen 30 percent over the next 1000 time periods is

\[
\left\{ \frac{S_{1000}}{S_0} \geq 1.3 \right\} = \left\{ \left( \frac{u}{d} \right)^{\sum_{i=1}^{1000} X_i} d^{1000} \geq 1.3 \right\}
\]

\[
= \left\{ \log(u/d) \sum_{i=1}^{1000} X_i + 1000 \log d \geq \log(1.3) \right\}
\]

\[
= \left\{ \sum_{i=1}^{1000} X_i \geq -1000 \log(d) + \log(1.3) \right\}
\]

\[
= \left\{ \frac{\sum_{i=1}^{1000} X_i - p}{\sqrt{1000 \log d (1 - p)}} \geq -1000 \log(d) - 1000p \log(u/d) + \log(1.3) \right\}
\]

But \( \sum_{i=1}^{1000} X_i \sim \text{Binom}(1000, 0.51) \), so, by the Central Limit Theorem, or the Normal approximation to the Binomial,

\[
P \left( \left\{ \frac{S_{1000}}{S_0} \geq 1.3 \right\} \right) \approx 1 - \Phi \left( \frac{-1000 \log(d) - 1000p \log(u/d) + \log(1.3)}{\log(u/d) \sqrt{1000 \sqrt{p(1 - p)}}} \right)
\]

\[
= 1 - \Phi(-2.58)
\]

\[
= 0.995.
\]
Q. 6) Let \((X_1, \ldots, X_n)\) be \(n\) independent random variables all with distribution \(\text{Exp}(1)\). Let 
\[
\hat{m}_n(X_1, \ldots, X_n) = \min_{1 \leq i \leq n} X_i.
\]
(a) What is the distribution of \(\hat{m}_n\)? (i.e. find the density or distribution function of \(\hat{m}_n\) and identify it, if possible)
(b) Compute \(E(\hat{m}_n)\).
(c) Compute \(\text{Var}(\hat{m}_n)\).

Solution:

(a) 
\[
P(\hat{m}_n(X_1, \ldots, X_n) > x) = P(\cap_{i=1}^{n} X_i > x) = \prod_{i=1}^{n} P(X_i > x) = (e^{-x})^n = e^{-nx}.
\]
Therefore, 
\(\hat{m}_n \sim \text{Exp}(n)\).

(b) As \(\hat{m}_n \sim \text{Exp}(n)\) 
\[E(\hat{m}_n) = \frac{1}{n}.\]

(c) As \(\hat{m}_n \sim \text{Exp}(n)\) 
\[\text{Var}(\hat{m}_n) = \frac{1}{n^2}.\]