Q. 1) **(Ross # 7.51)** The joint density of $X$ and $Y$ is given by

\[ f(x, y) = \begin{cases} \frac{x^n}{y} & 0 < x < y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases} \]

Compute $E[X^3 | Y = y]$.

**Solution:**

Since $f_X|Y(x|y) = \frac{e^{-y/X}y^x}{\Gamma(x) y^x} = \frac{1}{y}$ for $0 < x < y$, then $E[X^3 | Y = y] = \int_0^y x^3 \frac{1}{y} dx = y^3/4$.

Q. 2) (a) If $X \sim \text{Poisson}(\lambda)$, compute $M_X(t)$.

(b) Show that if $X_n \sim \text{Binomial}(n, \lambda/n)$ then $M_{X_n}(t)$ converges to $M_X(t)$ for every $t$, where $X \sim \text{Poisson}(\lambda)$.

**Solution:**

(a)

\[ M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = \exp(-\lambda + \lambda e^t) \]

(b)

\[ M_{X_n}(t) = E[e^{tX_n}] = \sum_{k=0}^{\infty} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = (pe^t+1-p)^n = \left(1 + \frac{\lambda}{n} (e^t-1)\right)^n \]

which in the limit, when $n$ goes to $\infty$, gives us $\exp(\lambda(e^t-1))$.

Q. 3) **(Ross # 8.9)** If $X$ is a Gamma random variable with parameters $(n, 1)$, approximately how large need $n$ be so that

\[ P \left[ \left| \frac{X}{n} - 1 \right| > 0.01 \right] < 0.01? \]

**Answer:**
(a) using Chebyshev’s inequality;
(b) the Central Limit Theorem.

Solution:

(a) Since \( E[X/n] = 1 \), by Chebyshev inequality we have:
\[
P\left(\left| \frac{X}{n} - 1 \right| > 0.01 \right) < \frac{\text{Var}(X/n)}{(0.01)^2} = 10^4/n
\]
Then \( 10^4/n < 0.01 \) if and only if \( n > 10^6 \).

(b) Let \( X_i \sim \text{Gamma}(1,1) \) for \( i = 1, \ldots, n \). Then \( X = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n,1) \)
\[
P\left(\left| \frac{X}{n} - 1 \right| > 0.01 \right) = P\left(\left| \frac{X_1 + X_2 + \ldots + X_n - n}{\sqrt{n}} \right| > 0.01\sqrt{n} \right) = P\left(\left| z \right| > 0.01\sqrt{n} \right)
\]
By central limit theorem \( z \) has standard normal distribution. So, look at the table to find that \( 0.01\sqrt{n} > 2.58 \), therefore \( n > (2.58)^2 \approx 66564 \).

Q. 4) (Ross # 8.11) Many people believe that the daily change of price of a company’s stock on the stock market is a random variable with mean 0 and variance \( \sigma^2 \). That is, if \( Y_n \) represents the price of the stock on the \( n \)-th day, then
\[
Y_n = Y_{n-1} + X_n, \quad n \geq 1
\]
where \( X_1, X_2, \ldots \) are independent and identically distributed random variables with mean 0 and variance \( \sigma^2 \). Suppose that the stock’s price today is 100. If \( \sigma^2 = 1 \), what can you say about the probability that the stock’s price will exceed 105 after 10 days, without using the Central Limit Theorem?

Solution:

Note, that \( Y_{10} - Y_0 = X_1 + X_2 + \ldots + X_{10} \), and \( E[Y_{10} - Y_0] = 0 \) as well as \( \text{Var}(Y_{10} - Y_0) = 10 \). Moreover, \( Y_{10} - Y_0 \) has normal distribution with the mean and variance above.

You can solve this problem by three different ways now.

1. Using the fact of standard normal distribution:
\[
P\{Y_{10} - Y_0 > 5\} = P\left\{ \frac{Y_{10} - Y_0}{\sqrt{10}} > \frac{5}{\sqrt{10}} \right\} \approx 0.057.
\]

2. Using ONE sided Chebyshev inequality:
\[
P\{Y_{10} - Y_0 > 5\} \leq \frac{\sigma^2}{\sigma^2 + a^2} = \frac{2}{7}.
\]
where $\sigma^2$ is the variance of $Y_{10} - Y_0$ and equals to 10. 3. Using TWO sided Chebyshev inequality:

\[
P\{Y_{10} - Y_0 > 5\} \leq P\{|Y_{10} - Y_0| > 5\} \\
\leq \frac{\text{Var} \left( \sum_{i=1}^{10} X_i \right)}{25} \\
= \frac{10}{25}.
\]