Q. 1) (Ross # 6.2) Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i$ equal 1 if the $i$-th ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

(a) $X_1, X_2$;
(b) $X_1, X_2, X_3$.

Solution:

(a) 

\[
\begin{align*}
P(X_1 = 1, X_2 = 1) &= \frac{5}{13} \times \frac{4}{12} \\
P(X_1 = 1, X_2 = 0) &= \frac{5}{13} \times \frac{8}{12} \\
P(X_1 = 0, X_2 = 1) &= \frac{8}{13} \times \frac{5}{12} \\
P(X_1 = 0, X_2 = 0) &= \frac{8}{13} \times \frac{7}{12}
\end{align*}
\]
Q. 2) (Ross # 6.6) A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by $N_1$ the number of tests made until the first defective is identified and by $N_2$ the number of additional tests until the second defective is identified: find the joint probability mass function of $N_1$ and $N_2$.

**Solution:**

Imagine drawing all 5 transistors and testing each one in order. Each outcome is equally likely and the number of outcomes is equal to the number of ways of choosing two “defective” transistors out of a set of 5, of which there are $\binom{5}{2} = 10$ ways. Hence the probability

$$P(N_1 = i, N_2 = j) = \frac{1}{10}, \quad 1 \leq i < j < 5.$$  

Q. 3) (Ross # 6.9) The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right), \quad 0 < x < 1, 0 < y < 2.$$  

(a) Verify that this is indeed a joint density function.
(b) Compute the density function of $X$.
(c) Find $P(X > Y)$.
(d) Find $P(Y > 1/2 | X < 1/2)$.
(e) Find $E(X)$.
(f) Find $E(Y)$.

**Solution:**

$$P(X_1 = 1, X_2 = 1, X_3 = 1) = \frac{5}{13} \frac{4}{12} \frac{3}{11}$$  

$$P(X_1 = 1, X_2 = 0, X_3 = 1) = \frac{7}{13} \frac{5}{12} \frac{4}{11}$$  

$$P(X_1 = 0, X_2 = 1, X_3 = 1) = \frac{8}{13} \frac{5}{12} \frac{4}{11}$$  

$$P(X_1 = 0, X_2 = 0, X_3 = 1) = \frac{8}{13} \frac{7}{12} \frac{5}{11}$$  

$$P(X_1 = 1, X_2 = 1, X_3 = 0) = \frac{5}{13} \frac{4}{12} \frac{8}{11}$$  

$$P(X_1 = 1, X_2 = 0, X_3 = 0) = \frac{5}{13} \frac{8}{12} \frac{7}{11}$$  

$$P(X_1 = 0, X_2 = 1, X_3 = 0) = \frac{8}{13} \frac{5}{12} \frac{7}{11}$$  

$$P(X_1 = 0, X_2 = 0, X_3 = 0) = \frac{8}{13} \frac{7}{12} \frac{6}{11}$$
(a) 
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = \int_{0}^{1} \int_{0}^{2} \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \, dy \, dx \\
= \int_{0}^{1} \left( \int_{0}^{2} \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \, dy \right) \, dx \\
= \frac{6}{7} \int_{0}^{1} \left( x^2 y + \frac{xy^2}{4} \right) \, dx \\
= \frac{6}{7} \int_{0}^{1} 2x^2 + x \, dx \\
= \frac{6}{7} \left( \frac{2x^3}{3} + \frac{x^2}{2} \right) \bigg|_{0}^{1} \\
= \frac{6}{7} \left( \frac{2}{3} + \frac{1}{2} \right) \\
= \frac{15}{56}.
\]

(b) From our above calculations
\[
f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy \\
= \int_{0}^{2} \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \, dy \\
= \frac{6}{7} \left( 2x^2 + x \right)
\]

(c) 
\[
P(X > Y) = \int_{(x,y) : x > y} f(x,y) \, dy \, dx \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x,y) \, dy \, dx \\
= \int_{0}^{1} \int_{0}^{x} x^2 + \frac{xy}{2} \, dy \, dx \\
= \frac{6}{7} \int_{0}^{1} \left( x^2 y + \frac{xy^2}{4} \right) \, dx \\
= \frac{6}{7} \int_{0}^{1} \left( x^3 + \frac{x^3}{4} \right) \, dx \\
= \frac{6}{7} \int_{0}^{1} x^3 \, dx \\
= \frac{6}{7} \left( \frac{1}{4} \right) \\
= \frac{15}{56}.
\]
(d)

\[ P(Y > 1/2 | X < 1/2) = \frac{P(Y > 1/2, X < 1/2)}{P(X < 1/2)} \]

\[ P(Y > 1/2, X < 1/2) = \frac{6}{7} \int_{0}^{1/2} \int_{1/2}^{2} x^2 + \frac{xy}{2} \, dy \, dx \]

\[ = \frac{6}{7} \int_{0}^{1/2} \left( x^2 y + \frac{xy^2}{4} \right) \bigg|_{1/2}^{2} \, dx \]

\[ = \frac{6}{7} \int_{0}^{1/2} \frac{3x^2}{2} + \frac{15x^2}{16} \, dx \]

\[ = \frac{6}{7} \left( \frac{x^3}{2} + \frac{15x^2}{32} \right) \bigg|_{0}^{1/2} \]

\[ = \frac{6}{7} \left( \frac{1}{16} + \frac{15}{128} \right) \]

\[ = \frac{6}{7} \frac{23}{128} \]

\[ = \frac{6}{7} \frac{23}{128} \]

\[ P(X < 1/2) = \frac{6}{7} \int_{0}^{1/2} 2x^2 + x \, dx \]

\[ = \frac{6}{7} \left( \frac{2x^3}{3} + \frac{x^2}{2} \right) \bigg|_{0}^{1/2} \]

\[ = \frac{6}{7} \left( \frac{1}{12} + \frac{1}{8} \right) \]

\[ = \frac{6}{7} \frac{5}{24} \]

\[ P(Y > 1/2 | X < 1/2) = \frac{23}{54} \approx 0.86. \]

(e)

\[ E(X) = \frac{6}{7} \int_{0}^{1} 2x^3 + x^2 \, dx = \frac{6}{7} \left( \frac{x^4}{2} + \frac{x^3}{3} \right) \bigg|_{0}^{1} = \frac{6}{7} \left( \frac{1}{2} + \frac{1}{3} \right) = \frac{6.5}{7} = \frac{5}{7}. \]
\[ E(Y) = \frac{6}{7} \int_0^1 \int_0^2 x^2 y + \frac{xy^2}{2} \, dy \, dx \]
\[ = \frac{6}{7} \int_0^1 \left( \frac{x^2 y^2}{2} + \frac{xy^2}{6} \right) \, dx \]
\[ = \frac{6}{7} \int_0^1 2x^2 + \frac{4x}{3} \, dx \]
\[ = \frac{6}{7} \left( \frac{2x^3}{3} + \frac{2x^2}{3} \right) \bigg|_0^1 \]
\[ = \frac{6}{7} \left( \frac{2}{3} + \frac{2}{3} \right) \]
\[ = \frac{6}{7} \cdot \frac{4}{3} = \frac{8}{7}. \]

Q. 4) (Ross # 6.20) The joint density of \(X\) and \(Y\) is given by

\[ f(x, y) = \begin{cases} 
xe^{-(x+y)} & x > 0, y > 0 \\
0 & \text{otherwise}.
\end{cases} \]

Are \(X\) and \(Y\) independent? What if \(f(x, y)\) were given by

\[ f(x, y) = \begin{cases} 
2 & 0 < x < y, 0 < y < 1 \\
0 & \text{otherwise}.
\end{cases} \]

Solution:
In the first case: yes, because

\[ f(x, y) = xe^{-x} 1_{[0, +\infty)}(x) \times ye^{-y} 1_{[0, +\infty)}(y). \]

In the second case: no, because the support of the random vector is

\[ \{(x, y) : 0 < x < y, 0 < y < 1\} \]

which is a triangle, and is not the Cartesian product of two sets. Hence \(X\) and \(Y\) cannot be independent.

Q. 5) (Ross # 6.29) When a current \(I\) (measured in amperes) flows through a resistance \(R\) (measured in ohms), the power generated is given by \(I^2R\) (measured in watts). Suppose that \(I\) and \(R\) are independent random variables with densities

\[ f_I(x) = 6x(1-x) \quad 0 \leq x \leq 1, \]
\[ f_R(x) = 2x \quad 0 \leq x \leq 1. \]
Determine the density function of $W$.

**Solution:**

First, we find the distribution function of $W$. For any distribution of $I$ and $R$ such that they are independent and non-negative

$$F_W(w) = P(W \leq w)$$
$$= P(I^2 R \leq w)$$
$$= P(\{(i,r) : i^2 r \leq w\})$$
$$= \int_0^\infty \int_0^{w/i^2} f_R(r)f_I(i) \, dr \, di$$
$$= \int_0^\infty f_I(i)F_R(w/i^2) \, di.$$

Therefore, the density

$$f_W(w) = \frac{d}{dw} \int_0^\infty f_I(i)F_R(w/i^2) \, di.$$
$$= \int_0^\infty f_I(i) \frac{1}{i^2} f_R(w/i^2) \, di.$$

In our particular example this boils down to, for $0 < w < 1$

$$f_W(w) = \int_{\sqrt{w}}^1 6i(1-i) \frac{1}{i^2} \frac{2w}{i^2} \, di.$$
$$= 12w \int_{\sqrt{w}}^1 \frac{1}{i^3} - \frac{1}{i^2} \, di$$
$$= 12w \left( \frac{1}{i} - \frac{1}{2i^2} \right)_{\sqrt{w}}^1$$
$$= 6w - 12w^{1/2} + 6.$$

The range of integration in the first line comes from the fact the density $f_R(w/i^2)$ is non-zero only if $w/i^2 \leq 1$. 

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