Q. 1) (Ross #5.19) Let $X$ be a normal random variable with mean 12 and variance 4. Find the value of $c$ such that $P(X > c) = 0.10$.

**Answer:** Letting $Z = (x - 12)/2$ then $Z$ is a standard normal. Now, $0.10 = P(Z > (c - 12)/2)$. From table 5.1, we find $P(Z < 1.28) = 0.90$ and so

$$(c - 12)/2 = 1.28 \quad \text{or} \quad c = 14.56$$

Q. 2) (Ross # 5.31)

(a) A fire station is to be located along a road of length $A$, $A < \infty$. If fires will occur at points uniformly chosen on $(0, A)$, where should the station be located so as to minimize the expected distance from the fire? That is, choose $a$ so as to minimize

$$E(|X - a|).$$

(b) Now suppose that the road is of infinite length – stretching from point 0 outward to $\infty$. If the distance of a fire from point 0 is exponentially distributed with rate $\lambda$, where should the fire now be located? That is, we want to minimize $E(|X - a|)$ where $X$ is now exponential with rate $\lambda$.

**Answer (a):**
Let the derivative be zero, we get \( \frac{2a}{A} - 1 = 0 \), thus \( a = \frac{A}{2} \). To check this is the minimizer, one can check the second derivative which is always positive, so it is indeed minimizer.

**Answer (b):**

\[
E[X - a] = \int_0^A |x - a| \frac{1}{A} dx \\
= \int_0^a (a - x) \frac{1}{A} dx + \int_a^A (x - a) \frac{1}{A} dx \\
= \frac{1}{A} \left( a^2 + A^2 - aA - \left( \frac{a^2}{2} - a^2 \right) \right) \\
= \frac{a^2}{A} - a + \frac{A}{2}
\]

Let the derivative be zero, we get \( \frac{2a}{A} - 1 = 0 \), thus \( a = \frac{A}{2} \). To check this is the minimizer, one can check the second derivative which is always positive, so it is indeed minimizer.

**Q. 3) (Ross #5.35)** The lung cancer hazard rate of a \( t \)-year-old male smoker, \( \lambda(t) \) is such that

\[
\lambda(t) = 0.027 + 0.00025 \cdot (t - 40)^2, \quad t \geq 40.
\]

Assuming that a 40-year-old male smoker survives all other hazards, what is the probability that he survives to

(a) age 50 without contracting lung cancer;
(b) age 60 without contracting lung cancer.

**Answer (a):** This problem is very similar to Example 5f of the textbook (pp216-217). The general result is as follows.
Suppose the hazard rate function is \( \lambda(t) \), then probability for an \( A \)-year person to survive until age \( B (B > A) \) is

\[
P\{ \text{A-year-old person reaches age } B \} = \frac{P\{ \text{his life time > } B \} \cdot P\{ \text{his life time > } A \}}{1 - F(B)/(1 - F(A))} = \exp\left( - \int_0^B \lambda(t)dt \right) / \exp\left( - \int_0^A \lambda(t)dt \right)
\]

Applying to this problem, the probability for him to survive to age 50 without contracting lung cancer is 
\[
\exp\left( - \int_0^{50} \lambda(t)dt \right) = e^{-353} = 0.7026
\]

**Answer (b):** Similarly, the probability for him to survive to age 60 without contracting lung cancer is 
\[
\exp\left( - \int_0^{60} \lambda(t)dt \right) = e^{-1.207} = 0.2991
\]

Q. 4) **(Ross #5.5 – Theoretical)** Use the fact that for any non-negative random variable \( Y \),

\[
E(Y) = \int_0^\infty P(Y > t) \, dt
\]

to prove that

\[
E(X^n) = \int_0^\infty nt^{n-1} P(X > t) \, dt.
\]

**Answer:**

\[
E[X^n] = \int_0^\infty P\{X^n > s\} \, ds = \int_0^\infty P\{X^n > t^n\} \, dt \quad \text{by} \quad s = t^n
\]

Q. 5) **(Ross # 5.28 Theoretical)** Let \( X \) be a continuous random variable having cumulative distribution function \( F \). Define a new random variable \( Y \) by \( Y = F(X) \). Show that \( Y \) is uniformly distributed over \((0,1)\).

**Answer:** Let \( F_Y(y) \) be the cumulative distribution function of \( Y \). It suffices to show \( F_Y(y) = y \) for \( y \in (0,1) \) since this is the cumulative distribution function for uniform random variable over \((0,1)\). In the following we use \( F_X(\cdot) \) to denote the cumulative distribution function of \( X \).
\[ F_Y(y) = P\{Y \leq y\} \]
\[ = P\{F_X(X) \leq y\} \]
\[ = P\{X \leq F_X^{-1}(y)\} \]
\[ = F_X(F_X^{-1}(y)) \]
\[ = y \]