Statistics 116 - Fall 2004
Theory of Probability
Assignment # 4
Due Monday, October 25

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Q. 1) (Ross #3.60) A true-false question is to be posed to a husband and a wife team on a quiz show. Both the husband and the wife will, independently, give the correct answer with probability $p$. Which of the following is a better strategy for the couple? i.e. Which strategy gives them a higher probability of correctly answering the question?

(a) Choose one of them by flipping a coin and let that person answer the question;

OR

(b) have them both consider the question and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give.

Answer: Both strategies have the same probability. Let $C = \text{answer is correct}$.

(a) Let $H = \text{husband answers}$, $W = \text{wife answers}$. Then $\mathbb{P}(C) = \mathbb{P}(C \cap H) + \mathbb{P}(C \cap W)$. Now, $\mathbb{P}(C \cap H) = \mathbb{P}(C) \mathbb{P}(H) = p \cdot \frac{1}{2}$. Similarly, $\mathbb{P}(C \cap W) = \frac{1}{2} p$. Hence $\mathbb{P}(C) = p$.

(b) Let $A = \text{both agree}$, $B = \text{they disagree}$. Then $\mathbb{P}(C) = \mathbb{P}(C \cap A) + \mathbb{P}(C \cap B)$. $\mathbb{P}(C \cap A) = p \cdot p = p^2$. And $\mathbb{P}(C \cap B) = 2 \cdot p \cdot (1 - p) \cdot \frac{1}{2} = p(1 - p)$. To see this, note that can break $C \cap B$ further as $(C \cap B \cap H) \cup (C \cap B \cap W)$. Now $C \cap B \cap H$ is the even that husband is correct, wife is incorrect and husband eventually answers (after coin flip). So, $\mathbb{P}(C) = p$.

Q. 2) (Ross #3.62) The probability of the closing of the $i$-th relay in the circuits shown is given by $p_i$, for $i = 1, \ldots, 5$. If all relays function independently, what is the probability that a current flows between $A$ and $B$ for the respective circuits in Figures 1 and 2?
Answer: Current passes if and only if at least one path is closed. A path is closed if all the relays along that path are closed.

(a) In Figure 1, let $E_1$ denote the event that circuit $(1,2,5)$ is closed. Let $E_2$ be the event that path $(3,4,5)$ is closed. So probability that current flows

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= P(1,2,5 \text{ closed}) + P(3,4,5 \text{ closed})$$

$$- P(1,2,3,4,5 \text{ closed})$$

$$= p_1 p_2 p_5 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5$$

(b) Define events $E_1 = (1,4)$ closed, $E_2 = (1,3,5)$ closed, $E_3 = (2,5)$ closed, and $E_4 = (2,3,4)$ closed. Then using the inclusion-exclusion formula again, and by direct computation as in (a), the probability
that current flows equals

\[ \mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) \]

\[ = \sum_{j=1}^{4} \mathbb{P}(E_j) - \sum_{j<k} \mathbb{P}(E_j \cap E_k) + \sum_{j<k<l} \mathbb{P}(E_j \cap E_k \cap E_l) \]

\[ - \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \]

\[ = p_1 p_4 + p_1 p_3 p_5 + p_2 p_5 + p_2 p_3 p_4 \]

\[ - p_1 p_3 p_4 p_5 - p_1 p_2 p_4 p_5 - p_1 p_2 p_3 p_4 - p_1 p_2 p_3 p_5 - p_2 p_3 p_4 p_5 \]

\[ - p_1 p_2 p_3 p_4 p_5 + 4 p_1 p_2 p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5 \]

Q. 3) **(Ross #4.21)** A total of 4 buses carrying 148 students from the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25 and 50 students. One of the students is randomly selected. Let \( X \) denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let \( Y \) denote the number of students on her bus.

(a) Which of \( E(X) \) or \( E(Y) \) do you think is larger? Why?

(b) Compute \( E(X) \) and \( E(Y) \).

**Answer :** Probability distribution of \( X \) is

\[ \mathbb{P}(X = 40) = \frac{40}{148}, \mathbb{P}(X = 33) = \frac{33}{148}, \mathbb{P}(X = 25) = \frac{25}{148}, \mathbb{P}(X = 50) = \frac{50}{148} \]

Probability distribution of \( Y \) is

\[ \mathbb{P}(Y = 40) = \mathbb{P}(Y = 33) = \mathbb{P}(Y = 25) = \mathbb{P}(Y = 50) = \frac{1}{4} \]

(a) \( E(X) \) seems to be bigger since it is more likely that a randomly selected student comes from a bus carrying largest number of students whereas a bus driver is equally likely to be from one of the four buses.

(b)

\[ E(X) = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} \approx 39.28 \]

\[ E(Y) = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37 \]

Q. 4) **(Ross #4.30)** A person tosses a fair coin until a tail appears for the first time. If the tail appears on the \( n \)-th flip, the person wins \( 2^n \) dollars. Let \( X \) denote the player’s winnings. Show that \( E(X) = +\infty \). This problem is known as the St. Petersburg’s paradox.

(a) Would be willing to pay $1 million to play this game once?
Would you be willing to pay $1 million for each game if you could play for as long as you like and only had to settle up when you stopped playing?

**Answer:**

\[ E(X) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = +\infty \]

(a) Since one gets to play only once, it is too risky to play. (Probability of winning $1 million or more is very small).

(b) If one plays an arbitrarily large number of times, the gains outweigh the costs and so it is safe to play.

**Q. 5** *(Ross #4.39)* A ball is drawn from an urn containing 3 white and 3 black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first 4 balls, exactly 2 are white?

**Answer:** Probability of success (a white ball is drawn) in each trial \( = p = \frac{3}{6} = \frac{1}{2} \). The trials are independent. So, # of successes in 4 trials is a Binomial(4, \( \frac{1}{2} \)) random variable. So,

\[ \mathbb{P}(\text{among first 4 balls 2 are white}) = \binom{4}{2} \left( \frac{1}{2} \right)^4 \]

**Q. 6** *(BONUS (10% – half of one question)) : (Ross #4.11)*

(a) An integer \( N \) is to be selected at random from \( \{ 1, \ldots , 10^3 \} \) in the sense that each integer has the same probability of being selected. What is the probability that \( N \) will be divisible by 3? by 5? by 7? by 105? How would your answer change if \( 10^3 \) is replaced by \( 10^k \) as \( k \) became larger and larger?

(b) For \( N \) to be chosen from \( \{ 1, \ldots , 10^k \} \) with \( k \) large as above, and any two fixed integers \( m_1 \) and \( m_2 \), what is the probability that \( N \) is divisible by \( m_1 \cdot m_2 \) as \( k \) gets larger and larger? What about any three fixed integers \( m_1, m_2, m_3 \)? What does this say about the events \( \{ N \text{ is divisible by } m \} \) for \( m = 1, 2, \ldots \), as \( k \) increases?

(c) An important function in number theory – one whose properties can be shown to be related to what is probably the most important unsolved problem of mathematics, the Riemann hypothesis – is the M"{o}bius function \( \mu(n) \), defined for all positive integral values \( n \) as follows. Factor \( n \) into its prime factors. If there is a repeated prime factor, as in \( 12 = 2 \cdot 2 \cdot 3 \) or \( 49 = 7 \cdot 7 \), then \( \mu(n) \) is defined to equal 0, otherwise, set \( \mu(n) = 1 \). Now let \( N \) be chosen at random from \( \{ 1, 2, \ldots , 10^k \} \), where \( k \) is large. Determine \( P\{ \mu(N) = 0 \} \) as \( k \to \infty \).
(Hint for (c): To compute \( P\{\mu(N) = 0\} \), use the identity
\[
\prod_{i=1}^{\infty} \left(1 - \frac{1}{P_i^2}\right) = \left(\frac{3}{4}\right) \left(\frac{8}{5}\right) \left(\frac{24}{25}\right) \left(\frac{48}{49}\right) \cdots = \frac{6}{\pi^2}
\]
where \( P_i \) is the \( i \)-th smallest prime. (We do not include 1 as a prime.)

Answer:

(a) \( P(\text{divisible by 3}) = \frac{333}{1000} \), \( P(\text{divisible by 5}) = \frac{200}{1000} \), \( P(\text{divisible by 7}) = \frac{142}{1000} \), \( P(\text{divisible by 105}) = \frac{9}{1000} \). In the limiting case as \( k \to \infty \), the probabilities converge to \( \frac{1}{3} \), \( \frac{1}{5} \), and \( \frac{1}{7} \), respectively.

(b) For \( k \) large, with \( N \) chosen uniformly from \( \{1, \ldots, 10^k\} \), \( P(N \text{ is divisible by } m_1 \cdot m_2 \cdot m_3) \) is approximately \( \frac{\text{LCM}(m_1, m_2, m_3)}{10^k} \) where LCM is the least common multiple. Therefore, if \( m_1, m_2, m_3 \) are relatively prime, this probability is approximately \( \frac{1}{m_1 m_2 m_3} \). This shows that the events \( P(N \text{ is divisible by } m_i, 1 \leq i \leq j) \) are approximately independent for fixed \( j \) and \( j \) relatively prime integers \( m_i \geq 1 \), for large \( k \).

(c) In the \( k \to \infty \) limit
\[
P(\mu(N) = 0) = P(N \text{ is not divisible by } P_i^2, i \geq 1)
= \prod_{i=1}^{\infty} \left(1 - \frac{1}{P_i^2}\right)^2 = \frac{6}{\pi^2}
\]