Q. 1) **(Ross #1.8 - Theoretical)** Prove that

\[
\binom{m+n}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}.
\]

**Hint:** Consider a group of \(n\) men and \(m\) women. How many groups of size \(r\) are possible?

**Solution:**

The left side is the total possibilities of drawing \(r\) individuals from the group of \(n\) men and \(m\) women. The \(k\)-th (\(0 \leq k \leq r\)) term on the right hand side gives the number of possibilities that among the \(r\) individuals drawn, \(k\) of them are men. By addition rule, the identity holds.

Q. 2) **(Ross #2.10)** Sixty percent of the students at certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) a ring or a necklace;

(b) a ring and a necklace?

**Solution:** Let \(A\) be the event that the student is wearing a ring and \(B\) be the event the student is wearing a necklace. Then, we know that

\[
P(A' \cap B') = 0.6
\]

\[
P(A) = 0.2
\]

\[
P(B) = 0.3
\]
We need to compute $P(A \cup B)$. 

$$P(A \cup B) = 1 - P(A^c \cap B^c) = 0.4$$

(b) We need to compute $P(A \cap B)$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.2 + 0.3 - 0.4 = 0.1.$$ 

Q. 3) **(Ross #2.45)** A woman has $n$ keys, of which one will open her door.

(a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her $k$-th try?

(b) What if she does not discard previously tried keys?

**Solution:**

(a) $P = \frac{n-1}{n} \times \cdots \times \frac{n-(k-1)}{n-(k-2)} \times \frac{1}{n-k+1} = \frac{1}{n}$

(b) $P = \left( \frac{n-1}{n} \right)^{k-1} \frac{1}{n}$

Q. 4) **(Ross #2.52)** A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be

(a) no complete pair;

(b) exactly one complete pair?

**Solution:**

(a) If are outcomes are unordered, there are $\binom{20}{8}$ possible outcomes. To have no pairs, our selection of 8 shoes must have 8 different colors: there are $\binom{10}{8}$ choices for the colors, and for each color there are 2 shoes of the same color in the closet. Therefore $P = \frac{\binom{10}{8} \binom{2}{8}}{\binom{20}{8}}$

(b) In this case, we need to choose 1 color for the pair, and in the remaining 6 shoes we must have 6 different colors (starting from 18 shoes because we have already chosen the pair). From (a) there are $\binom{10}{8} \binom{2}{8}$ ways of choosing the 6 non-matching shoes, and $\binom{10}{1}$ ways of choosing the pair. Therefore, $P = \frac{\binom{10}{1} \binom{9}{6} \binom{2}{8}}{\binom{20}{8}}$

Q. 5) **(Ross #2.20 - Theoretical)** Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have positive probability of occurring? Why or why not?

**Solution:**
We prove by contradiction. Suppose each element has equal positive probability \( p > 0 \). Since the total probability of the whole set is 1, we have contradiction:

\[
1 = \sum_{i=1}^{\infty} P(a_i) = \sum_{i=1}^{\infty} p = \infty
\]

Similarly, if \( p = 0 \), then

\[
\sum_{i=1}^{\infty} p = 0 \neq 1.
\]

Yes: all points can have positive probability. For example, let

\[
p_i = 2^{-i}, \quad i \geq 1.
\]

Then, for each \( i < \infty \), \( p_i > 0 \) and

\[
\sum_{i=1}^{\infty} = \frac{1}{2} \sum_{i=0}^{\infty} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}} = 1.
\]