Stable Super-resolution of Positive Sources

Veniamin I. Morgenshtern

Statistics Department, Stanford

Joint work with Emmanuel Candès
Diffraction limits resolution:

\[ \lambda_{\text{c}} = \frac{\lambda_{\text{LIGHT}}}{2 \sin(\theta)} \]

Ernst Abbe
Diffraction limits resolution: \( \lambda_c = \frac{\lambda_{\text{LIGHT}}}{2n \sin(\theta)} \)
Looking inside the cell: conventional microscopy

microtubule
Nobel Prize in Chemistry 2014

Eric Betzig  
Stefan W. Hell  
W.E. Moerner

Invention of single-molecule microscopy
Looking inside the cell

conventional microscopy  single-molecule microscopy
Structure of interest contains fluorescent molecules

All off
Structure of interest contains fluorescent molecules

All off

All on
Structure of interest contains fluorescent molecules

All off

All on

Detector

Cannot resolve the structure!
“Blinking” molecules: sparsity

Frame 1
“Blinking” molecules: sparsity

Frame 1
“Blinking” molecules: sparsity

Frame 1

Frame 2

Frame 3

Apply super-resolution algorithm.
Combine $\sim 10000$ frames.
The structure is now resolved!
“Blinking” molecules: sparsity

Frame 1  Frame 2  Frame 3

Apply super-resolution algorithm.
“Blinking” molecules: sparsity

Apply super-resolution algorithm.

Combine $\sim 10000$ frames.
“Blinking” molecules: sparsity

Apply super-resolution algorithm.

Combine $\sim 10000$ frames.

The structure is now resolved!
Next Frontier: image dynamical processes

How can we make data acquisition faster?

slow
super-resolution is easy

fast
super-resolution is difficult

Need powerful super-resolution algorithm!
Which algorithm?
Performance guarantees?
Fundamental limits?
Next Frontier: image dynamical processes

How can we make data acquisition faster?

slow
super-resolution is easy

fast
super-resolution is difficult

Need powerful super-resolution algorithm!
Next Frontier: image dynamical processes

How can we make data acquisition faster?

- slow
  super-resolution is easy

- fast
  super-resolution is difficult

Need powerful super-resolution algorithm!

Which algorithm?
Performance guarantees?
Fundamental limits?
Theory
Mathematical model (discrete 1D setup for simplicity)

Object

\[ x(t) = \sum_i x_i \delta(t - t_i), \quad x_i \geq 0 \]

Detector

\[ \lambda_c = 1/f_c \]

\[ s(t) = \int f_{\text{low}}(t - t')x(t')dt' \]

\[ f_{\text{low}}(t) = \frac{1}{2f_c} \left( \frac{\sin(2\pi f_c t)}{\pi t} \right)^2 \]
Mathematical model (discrete 1D setup for simplicity)

Object

\[ x(t) = \sum_i x_i \delta(t - t_i), \quad x_i \geq 0 \]

Detector

\[ \lambda_c = 1/f_c \]

\[ s(t) = \int f_{low}(t - t') x(t') dt' \]

\[ \mathbf{x} = [x_0 \cdots x_{N-1}]^T \geq 0 \]

\[ s = \mathbf{P} x + z, \quad \|z\|_1 \leq \delta \]

\[ \mathbf{P} = \mathbf{P}_{\text{tri}} \text{ is circulant} \]

Triangular spectrum

\[ -N/2 + 1 \quad -f_c \quad 0 \quad f_c \quad N/2 \]
Mathematical model (discrete 1D setup for simplicity)

Object

\[ x(t) = \sum_{i} x_i \delta(t - t_i), \quad x_i \geq 0 \]

Detector

\[ s(t) = \int f_{\text{low}}(t - t') x(t') \, dt' \]

\[ \mathbf{x} = [x_0 \cdots x_{N-1}]^T \geq 0 \]

\[ \mathbf{s} = \mathbf{P} \mathbf{x} + \mathbf{z}, \quad \|\mathbf{z}\|_1 \leq \delta \]

\[ \mathbf{P} = \mathbf{P}_{\text{flat}} \text{ is circulant} \]

Flat spectrum
Super-resolution factor and stability

\[ x = [x_0 \cdots x_{N-1}]^T \]

Triangular spectrum

\[
\begin{align*}
\mathbf{P} \mathbf{x} &+ \mathbf{z}, \\
\| \mathbf{z} \| &\leq \delta
\end{align*}
\]

Flat spectrum

\[ \text{SRF} \triangleq \frac{N}{2f_c} \]
Super-resolution factor and stability

\[ \mathbf{x} = [x_0 \cdots x_{N-1}]^T \]

Triangular spectrum

Flat spectrum

\[ s = \mathbf{P} \mathbf{x} + \mathbf{z}, \quad \| \mathbf{z} \| \leq \delta \]

SRF \( \triangleq \frac{N}{2f_c} \)

Stability: \( \| \mathbf{x} - \hat{\mathbf{x}} \| \overset{?}{\leq} \delta \cdot \text{(amplification factor)} \)
Rayleigh-regularity: $\mathbf{x} \in \mathcal{R}(d, r)$

$\mathbf{x}$ has fewer than $r$ spikes in every $\lambda_c d$ interval $[\lambda_c \triangleq 1/f_c]$
Rayleigh-regularity: \( x \in \mathcal{R}(d, r) \)

\( x \) has fewer than \( r \) spikes in every \( \lambda_c d \) interval \([\lambda_c \triangleq 1/f_c]\)

Separation: \( \mathcal{R}(2, 1) \)

\[ \geq 2\lambda_c \]
Rayleigh-regularity: $x \in \mathcal{R}(d, r)$

$x$ has fewer than $r$ spikes in every $\lambda_c d$ interval $[\lambda_c \triangleq 1/f_c]$
Rayleigh-regularity: $\mathbf{x} \in \mathcal{R}(d, r)$

$x$ has fewer than $r$ spikes in every $\lambda_c d$ interval $[\lambda_c \triangleq 1/f_c]$

**Separation:**

- $\mathcal{R}(2, 1)$
  - $\geq 2\lambda_c$
- $\mathcal{R}(4, 2)$
  - $\lambda_c \geq 4\lambda_c$
- $\mathcal{R}(6, 3)$
  - $\geq 6\lambda_c$
Key contribution

[\text{Prony’1795}]
\[ \mathbf{x} \in \mathbb{C}^N \]
no stability

\[ \geq 0 \]
efficient

\[ \text{Rayleigh-regularity} \]

\[ \text{convex} \]
## Key contribution

<table>
<thead>
<tr>
<th>[Prony’1795]</th>
<th>MUSIC, ESPRIT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in \mathbb{C}^N$</td>
<td>$x \in \mathbb{C}^N$</td>
</tr>
<tr>
<td>no stability</td>
<td>stability not understood</td>
</tr>
<tr>
<td>efficient</td>
<td>efficient</td>
</tr>
</tbody>
</table>

Rayleigh-regularity

Convex

Combinatorial

Efficient

Stability not understood
<table>
<thead>
<tr>
<th>[Prony’1795]</th>
<th>MUSIC, ESPRIT</th>
<th>[Donoho’92]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{x} \in \mathbb{C}^N$</td>
<td>$\mathbf{x} \in \mathbb{C}^N$</td>
<td>$\mathbf{x} \in \mathbb{C}^N$</td>
</tr>
<tr>
<td>no stability</td>
<td>stability not understood</td>
<td>stability</td>
</tr>
<tr>
<td>efficient</td>
<td>efficient</td>
<td>Rayleigh-regularity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>combinatorial</td>
</tr>
</tbody>
</table>

$\mathcal{R}(2r, r)$
Key contribution

[Prony’1795] \[x \in \mathbb{C}^N\]
no stability
- efficient

MUSIC, ESPRIT
\[x \in \mathbb{C}^N\]
stability not understood
- efficient

[Donoho’92] \[x \in \mathbb{C}^N\]
stability
Rayleigh-regularity
combinatorial

[Donoho et al.’90] \[x \geq 0\]
no stability
- convex

\[x \geq 0\]
Key contribution

[ Prony’1795 ]
\[ \mathbf{x} \in \mathbb{C}^N \]
no stability
- efficient

MUSIC, ESPRIT
\[ \mathbf{x} \in \mathbb{C}^N \]
stability not understood
- efficient

[ Donoho’92 ]
\[ \mathbf{x} \in \mathbb{C}^N \]
stability
Rayleigh-regularity
combinatorial

[ Donoho et al.’90 ]
\[ \mathbf{x} \geq 0 \]
no stability
- convex

[ Candès & F.-Granda’12 ]
\[ \mathbf{x} \in \mathbb{C}^N \]
stability
separation
convex

Works:

\[ \mathcal{R}(2, 1) \]

\[ \geq 2\lambda_c \]

0 \hspace{2cm} 1
Key contribution

- **[Prony’1795]**
  - $x \in \mathbb{C}^N$
  - no stability
  - efficient

- **MUSIC, ESPRIT**
  - $x \in \mathbb{C}^N$
  - stability not understood
  - efficient

- **[Donoho’92]**
  - $x \in \mathbb{C}^N$
  - stability
  - Rayleigh-regularity
  - combinatorial

- **[Donoho et al.’90]**
  - $x \geq 0$
  - no stability
  - convex

- **[Candès & F.-Granda’12]**
  - $x \in \mathbb{C}^N$
  - stability
  - separation
  - convex

Breaks:

$$\geq 4\lambda_c$$

$$\mathcal{R}(4, 2)$$
### Key contribution

<table>
<thead>
<tr>
<th>[Prony’1795]</th>
<th>MUSIC, ESPRIT</th>
<th>[Donoho’92]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in \mathbb{C}^N$</td>
<td>$x \in \mathbb{C}^N$</td>
<td>$x \in \mathbb{C}^N$</td>
</tr>
<tr>
<td>no stability</td>
<td>stability not understood</td>
<td>stability</td>
</tr>
<tr>
<td>–</td>
<td>–</td>
<td>Rayleigh-regularity</td>
</tr>
<tr>
<td>efficient</td>
<td>efficient</td>
<td>combinatorial</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>[Donoho et al.’90]</th>
<th>[Candès &amp; F.-Granda’12]</th>
<th>This work</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \geq 0$</td>
<td>$x \in \mathbb{C}^N$</td>
<td>$x \geq 0$</td>
</tr>
<tr>
<td>no stability</td>
<td>stability</td>
<td>stability</td>
</tr>
<tr>
<td>–</td>
<td>separation</td>
<td>Rayleigh-regularity</td>
</tr>
<tr>
<td>convex</td>
<td>convex</td>
<td>convex</td>
</tr>
</tbody>
</table>

$$\mathcal{R}(2r, r), x \geq 0$$
Main results

Recall:

\[ s = Px + z, \quad \|z\|_1 \leq \delta \]

Solve:

\[
\text{minimize} \quad \|s - P\hat{x}\|_1 \quad \text{subject to} \quad \hat{x} \geq 0
\]

Theorem: [V. Morgenshtern and E. Candès, 2014]

Take \( P = P_{\text{tri}} \) or \( P = P_{\text{flat}} \). Assume \( x \geq 0, x \in \mathcal{R}(2r, r) \). Then,

\[
\|\hat{x} - x\|_1 \leq c \cdot \delta \cdot \left( \frac{N}{2f_c} \right)^{2r}.
\]
Main results

Recall:

\[ s = P x + z, \quad \|z\|_1 \leq \delta \]

Solve:

\[
\begin{align*}
\text{minimize} & \quad \|s - P \hat{x}\|_1 \\
\text{subject to} & \quad \hat{x} \geq 0
\end{align*}
\]

Theorem: [V. Morgenshtern and E. Candès, 2014]

Take \( P = P_{\text{tri}} \) or \( P = P_{\text{flat}} \). Assume \( x \geq 0, \ x \in \mathcal{R}(2r, r) \). Then,

\[
\| \hat{x} - x \|_1 \leq c \cdot \delta \cdot \left( \frac{N}{2f_c} \right)^{2r}.
\]

Converse: [V. Morgenshtern and E. Candès, 2014]

For \( P = P_{\text{tri}} \), no algorithm can do better than \( c \cdot \delta \cdot \left( \frac{N}{2f_c} \right)^{2r - 1} \).
Key ideas

→ **Duality theory:** to prove stability we need a low-frequency trigonometric polynomial that is “curvy”
- [Dohono, et al.’92] construct trigonometric polynomial that is not “curvy”
- [Candès and Fernandez-Granda’12] construct trigonometric polynomial that is “curvy”, but construction needs separation
- **New construction:** multiply “curvy” trigonometric polynomials
  - “curvy”
  - construction needs no separation
Basic Lemma (duality in convex optimization)

- \( T \) is the support of \( x \)
Basic Lemma (duality in convex optimization)

- $\mathcal{T}$ is the support of $x$

- Suppose, we can construct a **low-frequency trig. polynomial**:
  \[
  q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt}, \quad 0 \leq q(t) \leq 1, \quad q(t_i) = 0 \text{ for all } t_i \in \mathcal{T}.
  \]
Basic Lemma (duality in convex optimization)

- $\mathcal{T}$ is the support of $x$
- Suppose, we can construct a **low-frequency trig. polynomial**:

$$ q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt}, \quad 0 \leq q(t) \leq 1, \quad q(t_i) = 0 \text{ for all } t_i \in \mathcal{T}. $$
Basic Lemma (duality in convex optimization)

- $\mathcal{T}$ is the support of $x$
- Suppose, we can construct a **low-frequency trig. polynomial**:
  \[
  q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt}, \quad 0 \leq q(t) \leq 1, \quad q(t_i) = 0 \text{ for all } t_i \in \mathcal{T}.
  \]

- Then, $\| \hat{x} - x \|_1 \leq 4\delta / \rho$. 
Proof of Lemma

Set: $h = \hat{x} - x$, $T = \{l/N : h_l < 0\}$
Proof of Lemma

Set: \( h = \hat{x} - x, \quad T = \{ l/N : h_l < 0 \} \subset \text{supp}(x). \)
Proof of Lemma

- Set: \( h = \hat{x} - x, \) \( \mathcal{T} = \{ l/N : h_l < 0 \} \subset \text{supp}(x). \)
- Dual vector \( q_l = q(l/N) \) satisfies:

\[
P_{\text{flat}} q = q, \quad \|q\|_\infty = 1, \quad \text{and} \quad \begin{cases} q_l = 0, & l/N \in \mathcal{T} \\ q_l > \rho, & \text{otherwise.} \end{cases}
\]
Proof of Lemma

- Set: \( h = \hat{x} - x \), \( \mathcal{T} = \{ l/N : h_l < 0 \} \subset \text{supp}(x) \).
- Dual vector \( q_l = q(l/N) \) satisfies:

\[
P_{\text{flat}}q = q, \quad \|q\|_\infty = 1, \quad \text{and} \quad \begin{cases} 
q_l = 0, & l/N \in \mathcal{T} \\
q_l > \rho, & \text{otherwise}.
\end{cases}
\]

- On the one hand:

\[
|\langle q - \rho/2, h \rangle| = |\langle P(q - \rho/2), h \rangle| = |\langle q - \rho/2, Ph \rangle| \\
\leq \|q - \rho/2\|_{\infty} \|Ph\|_1 \leq \|P\mathbf{x} - s + s - P\hat{x}\|_1 \\
\leq \|P\mathbf{x} - s\|_1 + \|s - P\hat{x}\|_1 \\
\leq 2\|\mathbf{P}\mathbf{x} - s\|_1 \leq 2\delta.
\]

\[\[\]\]
Proof of Lemma

- Set: \( h = \hat{x} - x \), \( \mathcal{T} = \{ l/N : h_l < 0 \} \subset \text{supp}(x) \).
- Dual vector \( q_l = q(l/N) \) satisfies:
  \[
P_{\text{flat}}q = q, \quad \|q\|_\infty = 1, \quad \text{and} \quad \begin{cases} q_l = 0, & l/N \in \mathcal{T} \\ q_l > \rho, & \text{otherwise}. \end{cases}
\]
- On the one hand:
  \[
  |\langle q - \rho/2, h \rangle| = |\langle P(q - \rho/2), h \rangle| = |\langle q - \rho/2, Ph \rangle| \\
  \leq \|q - \rho/2\|_\infty \|Ph\|_1 \leq \|Px - s + s - P\hat{x}\|_1 \\
  \leq \|Px - s\|_1 + \|s - P\hat{x}\|_1 \\
  \leq 2\|Px - s\|_1 \leq 2\delta.
\]
- On the other hand:
  \[
  |\langle q - \rho/2, h \rangle| = \left| \sum_{l=0}^{N-1} (q_l - \rho/2)h_l \right| = \sum_{l=0}^{N-1} (q_l - \rho/2)h_l \geq \rho\|h\|_1/2.
\]
Proof of Lemma

- Set: $h = \hat{x} - x$, $\mathcal{T} = \{l/N : h_l < 0\} \subset \text{supp}(x)$.
- Dual vector $q_l = q(l/N)$ satisfies:

$$P_{\text{flat}}q = q, \quad \|q\|_\infty = 1,$$

and

$$q_l = 0, \quad l/N \in \mathcal{T}$$

$$q_l > \rho, \quad \text{otherwise.}$$

- On the one hand:

$$|\langle q - \rho/2, h \rangle| = |\langle P(q - \rho/2), h \rangle| = |\langle q - \rho/2, Ph \rangle|$$

$$\leq \|q - \rho/2\|_\infty \|Ph\|_1 \leq \|Px - s + s - P\hat{x}\|_1$$

$$\leq \|Px - s\|_1 + \|s - P\hat{x}\|_1$$

$$\leq 2\|Px - s\|_1 \leq 2\delta.$$ 

- On the other hand:

$$|\langle q - \rho/2, h \rangle| = \left| \sum_{l=0}^{N-1} (q_l - \rho/2)h_l \right| = \sum_{l=0}^{N-1} (q_l - \rho/2)h_l \geq \rho \|h\|_1/2.$$ 

- Combining: $\|h\|_1 \leq 4\delta/\rho$. 
Key ideas

- **Duality theory**: to prove stability we need a low-frequency trigonometric polynomial that is “curvy”

  \[\text{Dohono, et al.'92}\] construct trigonometric polynomial that is not “curvy”

- **[Candès and Fernandez-Granda’12]** construct trigonometric polynomial that is “curvy”, but construction needs separation

- **New construction**: multiply “curvy” trigonometric polynomials
  - “curvy”
  - construction needs no separation
[Dohono, et al.’92]: “Classical” $q(t)$

\[ q(t) = \prod_{t_0 \in T} \frac{1}{2} \left[ \cos(2\pi(t + 1/2 - t_0)) + 1 \right]. \]

No separation required

Low curvature!

\[ q(t - t_0) \approx (t - t_0)^2 \Rightarrow \| x - \hat{x} \|_1 \leq \delta \cdot N^2 \]
- **Duality theory**: to prove stability we need a low-frequency trigonometric polynomial that is “curvy”

- [Dohono, et al.’92] construct trigonometric polynomial that is not “curvy”

→ [Candès and Fernandez-Granda’12] construct trigonometric polynomial that is “curvy”, but construction needs separation

- **New construction**: multiply “curvy” trigonometric polynomials
  - “curvy”
  - construction needs no separation
[Candès, Fernandez-Granda’12]: “Curvy” $q(t)$

$$q(t) = \sum_{t_j \in \mathcal{T}} a_j K(t - t_j) + \text{corrections},$$

$K(t)$... low-frequency and “curvy”

Separation between zeros required: $\mathcal{T} \in \mathcal{R}(2, 1)$

High curvature!

$$q(t - t_i) \approx f_c^2(t - t_i)^2 \Rightarrow \|x - \hat{x}\|_1 \leq c \cdot \delta \cdot \left(\frac{N}{2f_c}\right)^2$$

\[\rho \quad \begin{array}{cccc}
\hline
& & \hline
t_1 & \geq 2\lambda_c & t_2 & t_3
\end{array}\]
Comparison of Trigonometric Polynomials

\[ \frac{-1}{2}, \frac{1}{2} \]

"classical" \( q(t) \approx t^2 \)

"curvy" \( q(t) \approx f_c^2 t^2 \)
Key ideas

- **Duality theory**: to prove stability we need a low-frequency trigonometric polynomial that is “curvy”

- [Dohono, et al.’92] construct trigonometric polynomial that is not “curvy”

- [Candès and Fernandez-Granda’12] construct trigonometric polynomial that is “curvy”, but construction needs separation

→ **New construction**: multiply “curvy” trigonometric polynomials

  - “curvy”
  - construction needs no separation
New construction: curvature without separation

Partition support: $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2, \quad r = 2$

Regularity: $\mathcal{T} \in \mathcal{R}(2 \cdot 2, 2) \Rightarrow \mathcal{T}_i \in \mathcal{R}(4, 1)$

\[ q(t; f_c) = q_1(t; f_c/2) \times q_2(t; f_c/2) \]
New construction: curvature without separation

Partition support: \( \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2, \quad r = 2 \)

Regularity: \( \mathcal{T} \in \mathcal{R}(2 \cdot 2, 2) \Rightarrow \mathcal{T}_i \in \mathcal{R}(4, 1) \)

\[
q(t; f_c) = q_1(t; f_c/2) \times q_2(t; f_c/2)
\]

High curvature!

\[
q(t - t_i) \approx \frac{f_c^{2r}}{r^{2r}} (t - t_i)^{2r} \Rightarrow \|x - \hat{x}\|_1 \leq c \cdot \delta \cdot \left( \frac{N}{2f_c} \right)^{2r}
\]
Summation vs. multiplication

Remember: $q(t)$ must be frequency-limited to $f_c$!
Remember: $q(t)$ must be frequency-limited to $f_c$!

[Donoho, et.al.]:

$$q(t) = \prod_{t_j \in T} \frac{1}{2} \left[ \cos(2\pi(t + 1/2 - t_j)) + 1 \right]$$

frequency one
Remember: $q(t)$ must be frequency-limited to $f_c$!

[Donoho, et.al.]:

$$q(t) = \prod_{t_j \in T} \frac{1}{2} \left[ \cos(2\pi(t + 1/2 - t_j)) + 1 \right]$$

[Candès, Fernandez-Granda]:

$$q(t) = \sum_{t_j \in T} a_j K(t - t_j)$$
Summation vs. multiplication

Remember: $q(t)$ must be frequency-limited to $f_c$!

[Donoho, et.al.]:

$$q(t) = \prod_{t_j \in \mathcal{T}} \frac{1}{2} \left[ \cos(2\pi(t + 1/2 - t_j)) + 1 \right]$$

[Donoho, et.al.]:

$$q(t) = \sum_{t_j \in \mathcal{T}} a_j K(t - t_j)$$

[Candès, Fernandez-Granda]:

This work:

$$q(t) = \prod_{k=1}^{r} \sum_{t_{jk} \in \mathcal{T}_k} a_{jk} K(t - t_{jk})$$
Connections
Connection to Bernstein theorem

Consider: \( q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt} \) with \( \|q\|_\infty \leq 1 \)

Then: \( \|q'\|_\infty \leq 2f_c \)
Consider: $q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt}$ with $\|q\|_\infty \leq 1$

Then: $\|q'\|_\infty \leq 2f_c$

“Curvy” $q(t)$ has best possible curvature!
Connection to Bernstein theorem

Consider: \[ q(t) = \sum_{k=-f_c}^{f_c} \hat{q}_k e^{-i2\pi kt} \] with \( \| q \|_\infty \leq 1 \)

Then: \( \| q' \|_\infty \leq 2f_c \)

“Curvy” \( q(t) \) has best possible curvature!

Since

\[
\begin{align*}
q(t_i) &= 0 \\
q'(t_i) &= 0 \\
\| q \|_\infty &\leq 1
\end{align*}
\]

We conclude:

\[
\| q' \|_\infty \leq 2f_c \Rightarrow \| q'' \|_\infty \leq (2f_c)^2
\]

\[
\Rightarrow q(t - t_i) \leq (2f_c)^2(t - t_i)^2
\]

\[
\Rightarrow q(t_i + 1/N) \leq \frac{(2f_c)^2}{N^2} = \frac{1}{SRF^2}
\]
Complex vs. positive signals

Why do we need $x \geq 0$?

$x \geq 0$

Interpolate zero on supp. of $x$

$x \in \mathbb{C}^N$

Interpolate sign$(x)$ on supp. of $x$

Does not exist! (Bernstein Th.)
Continuous setup
$f_c$ fixed, $N \to \infty \Rightarrow SRF_{OLD} \to \infty$

Is the problem hopeless?

$\lambda_c s(t) = (f_{low} \star x)(t)$

$\lambda_{hi} \hat{x}(t) = (f_{hi} \star x)(t)$

Error = $\|f_{hi} \star (x - \hat{x})\|_1$

SRF_{NEW} = $\lambda_c / \lambda_{hi}$
$f_c$ fixed, $N \to \infty \Rightarrow SRF_{OLD} \to \infty$

Is the problem hopeless?

No: we need to be less ambitions!
\( f_c \) fixed, \( N \to \infty \Rightarrow \text{SRF}_{\text{OLD}} \to \infty \\

\[ x(t) \quad \text{and} \quad \hat{x}(t) \]

Is the problem hopeless?

No: we need to be less ambitious!

\[
\begin{align*}
\lambda_c &\quad \mathbf{s}(t) = (f_{\text{low}} \ast x)(t)
\end{align*}
\]

\[
\begin{align*}
\hat{\lambda}_c &\quad \hat{x}(t) = (f_{\text{hi}} \ast x)(t)
\end{align*}
\]
$f_c$ fixed, $N \to \infty \Rightarrow \text{SRF}_{\text{OLD}} \to \infty$

Is the problem hopeless?

No: we need to be less ambitions!

$s(t) = (f_{\text{low}} \star x)(t)$

$\hat{x}(t) = (f_{\text{hi}} \star x)(t)$

Error$=\|f_{\text{hi}} \star (x - \hat{x})\|_1$

$\text{SRF}_{\text{NEW}} = \frac{\lambda_c}{\lambda_{\text{hi}}}$
Theorem: [V. Morgenshtern and E. Candès, 2014]

Assume $x(t) \geq 0$, $x(t) \in \mathcal{R}(2r, r)$. Then,

$$\| f_{hi} \star (x - \hat{x}) \|_1 \leq c \cdot \left( \frac{\lambda_c}{\lambda_{hi}} \right)^{2r} \cdot \| z(t) \|_1.$$
Need new tools

**Theorem: [V. Morgenshtern and E. Candès, 2014]**

Assume $x(t) \geq 0$, $x(t) \in \mathcal{R}(2r, r)$. Then,

\[
\|f_{hi} \star (x - \hat{x})\|_1 \leq c \cdot \left( \frac{\lambda_c}{\lambda_{hi}} \right)^{2r} \cdot \|z(t)\|_1.
\]

**Can do:** all zeros

**Need:** arbitrary pattern $\{0, +\rho\}$
Control behavior on separated set

Multiply

\[ q(t) = q_1(t) \times q_2(t) \]

\[ 0 = q'(t_3) = q_1'(t_3)q_2(t_3) + q_1(t_3)q_2'(t_3) \]
New tools

1 Control behavior on separated set

2 Multiply

\[ q(t) = q_1(t) \times q_2(t) \]
\[ 0 = q'(t_3) = q_1'(t_3)q_2(t_3) + q_1(t_3)q_2'(t_3) \]

3 Sum

\[ q(t) = \sum_{r} \prod_{k=1}^{r} \sum_{t_{jk} \in T_k} a_{jk}K(t - t_{jk}) \]

\text{frequency } f_c/r
Theorem: [V. Morgenshtern and E. Candès, 2014]

Take \( P = P_{\text{tri},2D} \) or \( P = P_{\text{flat},2D} \). Assume \( x \geq 0 \), \( x \in \mathcal{R}(2.38r, r) \). Then,

\[
\| \hat{x} - x \|_1 \leq c \cdot \left( \frac{N}{2f_c} \right)^{2r} \delta.
\]

**New**: number of spikes is linear in the number of observations
Improving microscopes

Collaboration with Moerner Lab, C.A. Sing-Long, E. Candès
Reconstruction of 3D signals from 2D data

Double-helix PSF

Normal PSF

picture from [Pavani and Piston'08]

2D double-helix data
Reconstruction of 3D signals from 2D data

Double-helix PSF

Normal PSF

picture from [Pavani and Piston'08]

2D double-helix data

\[ \text{minimize } \| s - P\hat{x} \|_1 \quad \text{subject to } \hat{x} \geq 0 \]

\[ P \text{ contains double-helix PSF slices} \]
Preliminary result: 4 times faster than state-of-the-art

10000 CVX problems solved
TFOCS first order solver
millions of variables
Flexible framework: smooth background separation

minimize \[ \frac{1}{2} \left\| s - P(x + b) \right\|_2^2 + \lambda \sigma \left\| x \right\|_1 \]

subject to

- \( x \geq 0 \)
- \( b \) low freq. trig. polynomial (background)
Conclusion

Convex optimization is a near-optimal method for super-resolution of positive sources

- Flexibility and good practical performance
- Non-asymptotic precise stability bounds
- Rayleigh-regularity is fundamental: separation between spikes is only one part of the picture
Lots of questions remain

- What is the best regularizer in the presence of stochastic noise?
- Fast parallel solver exploiting the structure of the problem
- Theory for Double-Helix reconstruction: 3D signal from 2D observations
- ...

39 / 43
Lots of questions remain

- What is the best regularizer in the presence of stochastic noise?
- Fast parallel solver exploiting the structure of the problem
- Theory for Double-Helix reconstruction: 3D signal from 2D observations
- ...
- **Tractable near-optimal algorithm for complex-valued signals?**
Acknowledgements

- Swiss National Science Foundation Fellowship
- Theory: collaboration with E. Candès
- Motivation and applications: collaboration with W.E. Moerner, C.A. Sing-Long, M.D. Lew, A. Backer, S.J. Sahl
- Helpful discussions and related work: C. Fernandez-Granda, M. Soltanolkotabi, R. Heckel