Mathematicians are like Frenchmen; whatever you say to them, they translate it into their own language and forthwith it is something entirely different.

Goethe

1. Large Sample Confidence Intervals

• $100(1-\alpha)\%$ large-sample confidence interval for a population mean μ.

$$\left(\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \,,\ \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right),$$

where $z_{\frac{\alpha}{2}}$ is the constant such that $P(|Z| \le z_{\frac{\alpha}{2}}) = 1-\alpha$.
Notes

(a) The values of $z_{\frac{\alpha}{2}}$ for various confidence coefficients are readily obtained from the table of normal probabilities. In particular,

$100(1-\alpha)\%$	$z_{\frac{\alpha}{2}}$
90%	1.645
95%	1.96
99%	2.58

(b) When σ is unknown, and n is large (say, bigger than 30), then it is usual to replace σ in the confidence interval formula by its estimator s.

(c) The confidence interval result holds as long as X_i have finite variance, not just when the X_i's are normal. If, however, X_i is normal, $i = 1, \ldots, n$, then the confidence interval above is exact (i.e. has exact confidence coefficient $100(1-\alpha)\%$).

• $100(1-\alpha)\%$ large sample confidence interval for a difference in means $\mu_X - \mu_Y$.

$$\left((\overline{X} - \overline{Y}) - z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \,,\ (\overline{X} - \overline{Y}) + z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}} \right).$$

When the true variances σ_X^2 and σ_Y^2 are unknown and n_1 and n_2 are large, then s_X^2 and s_Y^2 are used in place of σ_X^2 and σ_Y^2, respectively, in the above formula.

• $100(1-\alpha)\%$ large sample confidence interval for a proportion p.

$$\left(\hat{p} - z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \,,\ \hat{p} + z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right).$$

• $100(1-\alpha)\%$ large sample confidence interval for a difference in proportions $p_1 - p_2$.

$$\left((\hat{p}_1 - \hat{p}_2) - z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \,,\ (\hat{p}_1 - \hat{p}_2) + z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \right).$$

2. Large sample hypothesis tests

Note: one-sided tests are given in parentheses ( ) and square brackets [ ]. These tests are valid for large n (say, $n > 30$), and population variances can be replaced by sample variances wherever appropriate.

• Large sample hypothesis test for a population mean μ. We wish to test $H_0 : \mu = \mu_0$ against the alternative $H_A : \mu \neq \mu_0$ ($H_A : \mu > \mu_0$) [$H_A : \mu < \mu_0$]. The test statistic is

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of Z either exceeds $z_{\frac{\alpha}{2}}$ or is smaller than $-z_{\frac{\alpha}{2}}$ (if the observed value of Z exceeds z_α) [if the observed value of Z is smaller than $-z_\alpha$].

• Large sample hypothesis test for a difference in means $\mu_X - \mu_Y$. We wish to test $H_0 : \mu_X - \mu_Y = D_0$ against the alternative $H_A : \mu_X - \mu_Y \neq D_0$ ($H_A : \mu_X - \mu_Y > D_0$) [$H_A : \mu_X - \mu_Y < D_0$]. The test statistic is

$$Z = \frac{\overline{X} - \overline{Y} - D_0}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of Z either exceeds $z_{\frac{\alpha}{2}}$ or is smaller than $-z_{\frac{\alpha}{2}}$ (if the observed value of Z exceeds z_α) [if the observed value of Z is smaller than $-z_\alpha$].

• Large sample hypothesis test for a proportion p. We wish to test $H_0 : p = p_0$ against the alternative $H_A : p \neq p_0$ ($H_A : p > p_0$) [$H_A : p < p_0$]. The test statistic is

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of Z either exceeds $z_{\frac{\alpha}{2}}$ or is smaller than $-z_{\frac{\alpha}{2}}$ (if the observed value of Z exceeds z_α) [if the observed value of Z is smaller than $-z_\alpha$].

• Large sample hypothesis test for a difference in proportions $p_1 - p_2$. We wish to test $H_0 : p_1 = p_2$ against the alternative $H_A : p_1 \neq p_2$ ($H_A : p_1 > p_2$) [$H_A : p_1 < p_2$]. Let $\hat{p} = (X+Y)/(n_1+n_2)$ denote the overall proportion of successes for the two populations. The test statistic is

$$Z = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of Z either exceeds $z_{\frac{\alpha}{2}}$ or is smaller than $-z_{\frac{\alpha}{2}}$ (if the observed value of Z exceeds z_α) [if the observed value of Z is smaller than $-z_\alpha$].

3. Small-sample confidence intervals

• $100(1-\alpha)\%$ small sample confidence interval for a population mean μ.

$$\left(\overline{X} - t_{(n-1)}(\alpha/2)\frac{s}{\sqrt{n}}\ ,\ \overline{X} + t_{(n-1)}(\alpha/2)\frac{s}{\sqrt{n}}\right),$$

where $t_{(n-1)}(\alpha/2)$ is the critical point of a $t_{(n-1)}$ distribution such that $P(|T| \leq t_{(n-1)}(\alpha/2)) = 1-\alpha$, i.e. $100(1-\alpha)\%$ of the area under the $t_{(n-1)}$ density lies between $-t_{(n-1)}(\alpha/2)$ and $t_{(n-1)}(\alpha/2)$.
• $100(1-\alpha)\%$ small sample confidence interval for a difference in means $\mu_X - \mu_Y$ from independent populations. To compute this interval we need to make the additional assumption that the variances of the two populations are equal (i.e. that $\sigma_X^2 = \sigma_Y^2$). Then, we estimate this common variance using the *pooled variance estimator*,

$$s_{\text{pooled}}^2 = \frac{(n_1-1)s_X^2 + (n_2-1)s_Y^2}{n_1+n_2-2}.$$

The confidence interval is then

$$\left(\overline{X} - \overline{Y} - t_{(n_1+n_2-2)}(\alpha/2)s_{\text{pooled}}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\ ,\ \overline{X} - \overline{Y} + t_{(n_1+n_2-2)}(\alpha/2)s_{\text{pooled}}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right).$$

Note that the degrees of freedom used for the t statistic is $n_1 + n_2 - 2$.
• $100(1-\alpha)\%$ small sample confidence interval for a difference in means $\mu_X - \mu_Y$ from paired data. The data is $(X_1, Y_1), \ldots, (X_n, Y_n)$. Form the differences $D_1 = X_1 - Y_1, \ldots, D_n = X_n - Y_n$, and let $\overline{D}$ and s_D denote the sample mean and sample standard deviation of the differences D_i, $i = 1, \ldots, n$. Then the interval is

$$\left(\overline{D} - t_{(n-1)}(\alpha/2)\frac{s_D}{\sqrt{n}}\ ,\ \overline{D} + t_{(n-1)}(\alpha/2)\frac{s_D}{\sqrt{n}}\right).$$

4. Small sample hypothesis tests

Note: One-sided tests are given in parentheses ( ) and square brackets [ ].

• Small sample hyptohesis test for a mean μ. We wish to test $H_0 : \mu = \mu_0$ against $H_A : \mu \neq \mu_0$ $(H_A : \mu > \mu_0)$ $[H_A : \mu < \mu_0]$. The test statistic is

$$T = \frac{\overline{X} - \mu_0}{s/\sqrt{n}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of T either exceeds $t_{(n-1)}(\alpha/2)$ or is smaller than $-t_{(n-1)}(\alpha/2)$ (if the observed value of T exceeds $t_{(n-1)}(\alpha)$) [if the observed value of T is smaller than $-t_{(n-1)}(\alpha)$]. Note that the $\alpha/2$'th critical point is used for two-sided tests while the α'th critical point is used for one-sided tests.

• Small sample hypothesis test for a difference in means $\mu_X - \mu_Y$ from independent populations. We wish to test $H_0 : \mu_X - \mu_Y = D_0$ against $H_A : \mu_X - \mu_Y \neq D_0$ ($H_A : \mu_X - \mu_Y > D_0$) [$H_A : \mu_X - \mu_Y < D_0$]. The test statistic is

$$T = \frac{\overline{X} - \overline{Y} - D_0}{s_{\text{pooled}}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},$$

where s^2_{pooled} is the pooled variance estimator $s^2_{\text{pooled}} = ((n_1 - 1)s^2_X + (n_2 - 1)s^2_Y)/(n_1 + n_2 - 2)$. We reject H_0 in favor of H_A at significance level α if the observed value of T e ither exceeds $t_{(n_1+n_2-2)}(\alpha/2)$ or is smaller than $-t_{(n_1+n_2-2)}(\alpha/2)$ (if the observed value of T exceeds $t_{(n_1+n_2-2)}(\alpha)$) [if the observed value of T is smaller than $-t_{n_1+n_2-2)}(\alpha)$]. Note the degrees of freedom for this test is $n_1 + n_2 - 2$ **not** $n - 1$.

• Small sample hypothesis test for a difference in means $\mu_X - \mu_Y$ from paired data. We are interested in testing $H_0 : \mu_X - \mu_Y = 0$ against $H_A : \mu_X - \mu_Y \neq 0$ ($H_A : \mu_X - \mu_Y > 0$) [$H_A : \mu_X - \mu_Y < 0$]. The test is based on the differences $D_i = X_i - Y_i$, $i = 1, \ldots, n$, so is equivalent to testing $H_0 : \mu_D = 0$ against $H_A : \mu_D \neq 0$ ($H_A : \mu_D > 0$) [$H_A : \mu_D < 0$], where μ_D is the mean of the population of differences. This is just a one-sample test for the mean μ_D, so the test statistic is

$$T = \frac{\overline{D} - 0}{s_D/\sqrt{n}}.$$

We reject H_0 in favor of H_A at significance level α if the observed value of T e ither exceeds $t_{(n-1)}(\alpha/2)$ or is smaller than $-t_{(n-1)}(\alpha/2)$ (if the observed value of T exceeds $t_{(n-1)}(\alpha)$) [if the observed value of T is smaller than $-t_{(n-1)}(\alpha)$].

5. Tests and confidence intervals for the variance

• $100(1 - \alpha)\%$ confidence interval for the population variance σ^2.

$$\left(\frac{(n-1)s^2}{\chi^2_{(n-1)}(\alpha/2)}\ ,\ \frac{(n-1)s^2}{\chi^2_{(n-1)}(1-\alpha/2)}\right),$$

where $\chi^2_{(n-1)}(\alpha/2)$ is the critical point of the $\chi^2_{(n-1)}$ distribution such that if $X \sim \chi^2_{(n-1)}$ then $P(X \leq \chi^2_{(n-1)}(\alpha/2)) = 1 - \alpha/2$.

• Hypothesis test for the variance σ^2. We wish to test the hypothesis $H_0 : \sigma^2 = \sigma^2_0$ against $H_A : \sigma^2 \neq \sigma^2_0$ ($H_A : \sigma^2 > \sigma^2_0$) [$H_A : \sigma^2 < \sigma^2_0$]. The test statistic is

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2_0}.$$

We reject H_0 in favor of H_A if the observed χ^2 value either exceeds $\chi^2_{(n-1)}(\alpha/2)$ or is smaller than $\chi^2_{(n-1)}(1-\alpha/2)$ (if the observed value of χ^2 exceeds $\chi^2_{(n-1)}(\alpha)$) [if the observed value of χ^2 is smaller than $\chi^2_{(n-1)}(1-\alpha)$]. Note that the $\alpha/2$'th critical point of the $\chi^2_{(n-1)}$ distribution is used for the two-tailed test, while the α'th critical point is used for the one-sided test.

• Hypothesis test for the equality of variances. Recall that in order to carry out a test of the hypothesis that $\mu_X = \mu_Y$ from independent populations, it was necessary to assume $\sigma_X^2 = \sigma_Y^2$. We can formally test this assumption as follows: we wish to test $H_0 : \sigma_X^2 = \sigma_Y^2$ against $H_A : \sigma_X^2 \neq \sigma_Y^2$ ($H_A : \sigma_X^2 > \sigma_Y^2$) [$H_A : \sigma_X^2 > \sigma_Y^2$]. The test statistic is

$$F = \frac{s_X^2}{s_Y^2}.$$

We reject H_0 in favor of H_A at significance level α if the observed F value exceeds $F_{n_1-1,n_2-1}(\alpha/2)$, the upper $\alpha/2$'th critical point of an F_{n_1-1,n_2-1} distribution, is falls below $F_{n_1-1,n_2-1}(1-\alpha/2)$ (if the observed value of F exceeds $F_{n_1-1,n_2-1}(\alpha)$) [if the observed value of F is smaller than $F_{n_1-1,n_2-1}(1-\alpha)$).

Tables for both the F and χ^2 distributions for various degrees of freedom are given at the back of the book. Note that in order to conduct some of the tests above we need use the special identity for F distributions:

$$F_{n,m}(1-\alpha) = \frac{1}{F_{m,n}(\alpha)}.$$