7.3 Lecture 8 Friday 01/02/01

Homework: New series due on 11th february.
See the logictics section

7.4 Review of last time, if you missed see: Lecture 7

We have considered simple random sample for the estimation of a parameter μ.

We obtained a random variable $\bar{X}$ which estimates μ,

which is an example of the general case where $\hat{\theta}$ estimates θ and we evaluate its precision by looking at the $SE(\hat{\theta})$ which is just another name for the standard deviation of the random variable $\hat{\theta}$.

Often by linearization or just because it is a sum of iid components :

$$\hat{\theta} \sim \mathcal{N}$$

this enables confidence statements, however for them to be useful we have to estimate $\hat{\theta}$'s SE, this is done using $s^2_{\hat{\theta}}$ and thus we produce confidence intervals for θ :

$$\hat{\theta} \pm s_{\hat{\theta}} \times z_{\alpha/2} \text{ is a } (1-\alpha)100\% \text{ confidence interval}$$

8 Parameter Estimation

8.1 Motivation

Fitting probability distributions to data.

$\boxed{\text{Why Fit Models ?}}$

- Predictive purposes
- Descriptive purposes
- Explicative purposes
- Simulation of complex phenomena

Often to describe an observed data set one can use a probability density, recognising which family it belongs to will be a matter of training and we will see about that later in the course.

For the time being suppose we "see" that it should be Normal, Poisson, Exponenital or Gamma, ...

This is a generalization of what we saw in the last chapter, here the parent unknown population is not a finite population but a probability distribution function with a few unknown parameters, (finitely many for a parametric situation).

That is the motivation for the estimation problems we are going to address.

9 The Method of Moments

We recall the definition of the kth moment of the distribution of teh random variable X is:

$$\mu_k = E(X^k)$$

if it exists

Suppose $X_1, X_2, \dots X_n$ iid from that distribution then the estimate for μ_k will be :

$$\hat{\mu_k} = \frac{1}{n}\sum_{i=1}^{n} X_i^k$$

The method of moments is based on the assumption that the parameters we want to estimate θ_1, θ_2 can be written as functions of the moments, maybe the two first but it could be more.

So if we can write:

$$\theta_1 = f_1(\mu_1, \mu_2) \qquad\qquad \theta 2 = f_2(\mu_1, \mu_2)$$

then the natural estimates will be obtained by just plugging in the estimates of μ_1 and μ_2:

$$\hat{\theta}_1 = f_1(\hat{\mu}_1, \hat{\mu}_2) \qquad\qquad \hat{\theta}_2 = f_2(\hat{\mu}_1, \hat{\mu}_2)$$

A simple example is that of fitting a Poisson distribution because it only has one parameter to specify it.

9.1 Moment Generating Function

Some extra information about mgf's.

Definition:

$$M(t) = E(e^{tX})$$

in the discrete case:

$$M(t) = \sum_x e^{tx} p(x)$$

and in the continuous case:

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx$$

If the mgf exists for t in an open interval around 0, it uniquely determines the probability distribution.

The rth moment of the pdf is $E(X^r)$ if this moment exists, which it doesn't always.

What has this to do with the moments?

$$M'(t) = \frac{d}{dt}\int e^{tx}f(x) = \int xe^{tx}f(x)dx$$

$$M'(0) = \int xf(x)dx = E(X)$$

and so on

$$M^{(r)}(0) = E(X^r)$$

Poisson Distribution:

$$M(t) = \sum_{k=0}^{\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

The derivatives are:

$$M'(t) = \lambda e^t e^{\lambda(e^t-1)}$$

and

$$M''(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t}e^{\lambda(e^t-1)}$$

and evaluation at 0 gives:

$$E(X) = \lambda, \qquad E(X^2) = \lambda + \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda$$

Gamma Distribution:

$$\begin{aligned}
M(t) &= \int_0^\infty e^{tx}\, x^{\alpha-1}e^{-\lambda x}dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{x(t-\lambda)} \\
\text{But } \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} &= \int_0^\infty x^{\alpha-1}e^{(\lambda-t)x}dx \\
M(t) &= (\frac{\lambda}{(\lambda-t)})^\alpha \\
M'(t) &= \lambda^\alpha\alpha(\lambda-t)^{-(\alpha+1)} \qquad M'(0) = \frac{\alpha}{\lambda} \\
M''(t) &= \lambda^\alpha\alpha(\alpha+1)(\lambda-t)^{-(\alpha+2)} \qquad M''(0) = \frac{\alpha(\alpha+1)}{\lambda^2}
\end{aligned}$$