9.4 Reminder about some of the pivotal distributions

See chapter 6 for more details.

■

1. When $Z \sim \mathcal{N}(0,1)$, $X = Z^2 \sim \chi^2_1$, differentiate the cdf

$$F(x) = P(X < x) = P(-\sqrt{x} < Z < \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

to obtain the density:

$$f(x) = \frac{1}{2}x^{-\frac{1}{2}}\phi(\sqrt{x}) + \frac{1}{2}x^{-\frac{1}{2}}\phi(-\sqrt{x}) = x^{-\frac{1}{2}}\phi(\sqrt{x}) = \frac{x^{-\frac{1}{2}}}{\sqrt{2\pi}}e^{-x/2},$$

$$x \geq 0, \qquad f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, \text{ with } \alpha = \frac{1}{2}, \lambda = \frac{1}{2}$$

(see thus that $\Gamma(\frac{1}{2}) = \int_0^\infty u^{-1/2}e^{-u}du = \sqrt{\pi}$).

2. When $Z_i \sim \mathcal{N}(0,1)$, and are independent, for $i = 1, 2, \ldots, m$ then

$$K = Z_1^2 + Z_2^2 + \ldots + Z_{m-1}^2 + Z_m^2 \sim \chi^2_m$$

This is the Gamma distribution with $\alpha = \frac{m}{2}$ and $\lambda = 1/2$
$E(K) = m, var(K) = 2m$.

3. The ratio of two independent χ^2 random variables divided by their respective degrees of freedom,is called an F random variable, with parameters the degrees of freedom of the numerator m and the denominator n: $F_{m,n}$.

$$W = \frac{U/m}{V/n} \sim F_{m,n}$$

4. Definition:
If $Z \sim \mathcal{N}(0,1)$ and $U \sim \chi^2_m$ are independent, the ratio

$$\frac{Z}{\sqrt{U/m}} \sim t_m$$

is called a t distribution with m degrees of freedom.