Statistics 200

Computer Lab Section #3

The aim of this exercise is to construct a minimum variance unbiased estimate for the proportion π of a Normal population that is above a given value c on the basis of an observed sample : $x_1, x_2, \ldots, x_n$.

1. Recall that $\bar{X}$ and the unbiased estimate of variance S^2 are sufficient for μ and σ^2.

2. Consider the statistic W defined by :

$$W = \begin{cases} 0 & if \quad x_1 \le c \\ 1 & if \quad x_1 > c \end{cases}$$

   Where x_1 is the first observation. It is easy to see that that W is an unbiased estimate of π.

3. Use the two following facts to write out how to implement the "Rao-Blackwellisation" estimator based on W and the sufficient statistics $\bar{X}$ and S.

   (a) Suppose $U \sim \Gamma(\alpha, p_1)$ and $V \sim \Gamma(\alpha, p_2)$ are independent gamma random variables, then $\frac{U}{U+V}$ is a r.v. which is independent of $U+V$ and has a density $b(x)$ proportional to $x^{p_1-1}(1-x)^{p_2-1}$.

   (b) The Beta distribution with parameters γ and δ is the distribution defined by the density :

$$b(x|\gamma, \delta) = [\beta(\gamma, \delta)]^{-1} x^{\gamma-1}(1-x)^{\delta-1}$$

   where $\beta(\gamma, \delta) = \Gamma(\gamma)\Gamma(\delta)/\Gamma(\gamma+\delta)$ is a normalizing constant to make it a density.

Question 1 *Estimate the mean and the variance of Beta*$(0.5, 6.5)$ *using the* **betarnd** *function.*

Question 2 *Calculate* $P(B \le 0.02)$ *with* $B \sim Beta(0.5, 6.5)$ *using the* **betacdf** *function.*

The minimum variance unbiased estimate for π is

$$\begin{aligned} \mathrm{E}(W|\bar{X}, S) &= P(X_1 > c|\bar{X}, S) \\ &= P\left(\frac{X_1 - \bar{X}}{S} > \frac{c - \bar{X}}{S} |\bar{X}, S\right) \\ &= P\left(\frac{X_1 - \bar{X}}{S} > \frac{c - \bar{X}}{S}\right) \qquad \text{(Basu's theorem)} \end{aligned}$$

Consider the distribution of

$$Y \equiv \frac{\frac{n}{n-1}(X_1 - \bar{X})^2}{(n-1)S^2}.$$

Use the facts that $X_1 - \bar{X} \sim N(0, \frac{n-1}{n}\sigma^2)$, $(n-1)S^2/\sigma^2 - \frac{n}{n-1}(X_1 - \bar{X})^2/\sigma^2 \sim \chi^2_{n-2}$ and they are independent of each other.

Question 3 *What is the distribution of Y?*

Note that the χ^2_n distribution is a special case of the gamma distribution, with $\alpha = 1/2$ and $p = n/2$.

Question 4 *What is $E(W|\bar{X}, S)$?*

Note that the distribution of $\frac{\sqrt{\frac{n}{n-1}}(X_1-\bar{X})}{\sqrt{(n-1)}S}$ is symmetric.

Question 5 *Write a Matlab function to calculate the minimum variance unbiased estimate for π.*

Question 6 *Obtain the estimate of π for $c = 4.8$ obtained by this method applied to the sample*

$$3.15, 2.92, 4.59, 6.48, 4.28, 4.81, 3.36, 3.20, 4.16, 6.48, 3.85, 3.72, 5.15, 2.67, 2.11$$

(For these numbers, $\bar{X} = 4.06, S^2 = 1.647$)