Example of Doing Two way ANOVA

1 Two Way Analysis of Variance by Hand

Error Decomposition

\[
SS_{total} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{r} (Y_{ijk} - \bar{Y}_{ij.} - \bar{Y}_{i.} - \bar{Y}_{..})^2
\]

\[
SS_A = r \cdot b \cdot \sum_{i=1}^{a} (Y_{i.} - \bar{Y}_{ij.} - \bar{Y}_{.j} - \bar{Y}_{..})^2
\]

\[
SS_B = r \cdot a \cdot \sum_{j=1}^{b} (Y_{.j} - \bar{Y}_{ij.} - \bar{Y}_{ij.} - \bar{Y}_{..})^2
\]

\[
SS_{A \times B} = (a-1)(b-1) \cdot \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij} - \bar{Y}_{ij.} - \bar{Y}_{i.} - \bar{Y}_{..})^2
\]

\[
SS_{within} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{r} (Y_{ijk} - \bar{Y}_{ij.})^2
\]

ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
<th>Degrees of Freedom</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>a-1</td>
<td>SS_A</td>
<td>MS_A</td>
<td>MS_A/MS_within</td>
</tr>
<tr>
<td>B</td>
<td>b-1</td>
<td>SS_B</td>
<td>MS_B</td>
<td>MS_B/MS_within</td>
</tr>
<tr>
<td>A × B</td>
<td>(a-1)(b-1)</td>
<td>SS_{A×B}</td>
<td>MS_{A×B}</td>
<td>MS_{A×B}/MS_within</td>
</tr>
<tr>
<td>Within</td>
<td>ab(r-1)</td>
<td>SS_{within}</td>
<td>MS_{within}</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>ab(r-1)</td>
<td>SS_{total}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example Suppose you want to determine whether the brand of laundry detergent used and the temperature affects the amount of dirt removed from your laundry. To this end, you buy two different brand of detergent (“Super” and “Best”) and choose three different temperature levels (“cold”, “warm”, and “hot”). Then you divide your laundry randomly into 6 × r piles of equal size and assign each r piles into the combination of (“Super” and “Best”) and (“cold”, “warm”, and “hot”). In this example, we are interested in testing Null Hypotheses

\[ H_{0_D} : \] The amount of dirt removed does not depend on the type of detergent

\[ H_{0_T} : \] The amount of dirt removed does not depend on the temperature

One says the experiment has two factors (Factor Detergent, Factor Temperature) at \(a = 2\) (Super and Best) and \(b = 3\) (cold, warm and hot) levels. Thus there are \(ab = 3 \times 2 = 6\) different combinations of detergent and temperature. With each combination you wash \(r = 4\) loads. \(r\) is called the number of replicates. This sums up to \(n = abr = 24\) loads in total. The amounts \(Y_{ijk}\) of dirt removed when washing sub pile \(k\) (\(k = 1, 2, 3, 4\)) with detergent \(i\) (\(i = 1, 2\)) at temperature \(j\) (\(j = 1, 2, 3\)) are recorded in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Cold</th>
<th>Warm</th>
<th>Hot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Super</td>
<td>4,5,6,5</td>
<td>7,9,8,12</td>
<td>10,12,11,9</td>
</tr>
<tr>
<td>Best</td>
<td>6,6,4,4</td>
<td>13,15,12,12</td>
<td>12,13,10,13</td>
</tr>
</tbody>
</table>

Solution:

<table>
<thead>
<tr>
<th></th>
<th>Cold</th>
<th>Warm</th>
<th>Hot</th>
<th>(m_D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Super</td>
<td>4,5,6,5 (5)</td>
<td>7,9,8,12 (9)</td>
<td>10,12,11,9 (10)</td>
<td>8</td>
</tr>
<tr>
<td>Best</td>
<td>6,6,4,4 (5)</td>
<td>13,15,12,12 (13)</td>
<td>12,13,10,13 (12)</td>
<td>10</td>
</tr>
<tr>
<td>(m_T)</td>
<td>5</td>
<td>11</td>
<td>11</td>
<td>9</td>
</tr>
</tbody>
</table>

- \(SS_{within}\) and \(df_{within}\)

\[
SS_{within} = \sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{k=1}^{4} (Y_{ijk} - \bar{Y}_{ij.})^2
\]
\[
(4 - 5)^2 + (5 - 5)^2 + (6 - 5)^2 + (5 - 5)^2 \\
+ (7 - 9)^2 + (9 - 9)^2 + (8 - 9)^2 + (12 - 9)^2 \\
\ldots \ldots \\
+ (12 - 12)^2 + (13 - 12)^2 + (10 - 12)^2 + (13 - 12)^2
\]
\[
= 38
\]
\[
df_{\text{within}} = (r - 1) \ast a \ast b = 3 \ast 2 \ast 3 = 18
\]
\[
MS_{\text{within}} = SS_{\text{within}} / df_{\text{within}} = 38/18 = 2.1111
\]

• \(SS_{\text{detergent}}\) and \(df_{\text{detergent}}\)

\[
SS_{\text{detergent}} = r \cdot b \cdot \sum_{i=1}^{2} \left( \bar{Y}_{i..} - \bar{Y}_{..} \right)^2
\]
\[
= 4 \times 3 \times \left( (8 - 9)^2 + (10 - 9)^2 \right) = 24
\]
\[
df_{\text{detergent}} = a - 1 = 1
\]
\[
MS_{\text{detergent}} = SS_{\text{detergent}} / df_{\text{detergent}} = 24 / 1 = 24
\]

• \(SS_{\text{temperature}}\) and \(df_{\text{temperature}}\)

\[
SS_{\text{temperature}} = r \cdot a \cdot \sum_{j=1}^{3} \left( \bar{Y}_{.j} - \bar{Y}_{..} \right)^2
\]
\[
= 4 \times 2 \times \left( (5 - 9)^2 + (11 - 9)^2 + (11 - 9)^2 \right) = 24
\]
\[
df_{\text{temperature}} = b - 1 = 2
\]
\[
MS_{\text{temperature}} = SS_{\text{temperature}} / df_{\text{temperature}} = 192 / 2 = 81
\]

• \(SS_{\text{interaction}}\) and \(df_{\text{interaction}}\)

\[
SS_{\text{interaction}} = r \times \sum_{i=1}^{2} \sum_{j=1}^{3} \left( \bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{.j} + \bar{Y}_{..} \right)^2
\]
\[
= 4 \times \left[ (5 - 8 - 5 + 9)^2 + (9 - 8 - 11 + 9)^2 + (110 - 8 - 11 + 9)^2 + \cdots + (12 - 11 - 10 + 9)^2 \right] = 12
\]
\[
df_{\text{interaction}} = (a - 1) \times (b - 1) = 2 \times 1 = 2
\]
\[
MS_{\text{interaction}} = SS_{\text{interaction}} / df_{\text{interaction}} = 12 / 2 = 6
\]

• F-Test

\[
MS_{\text{detergent}} / MS_{\text{within}} \sim F(df_{\text{detergent}}, df_{\text{within}})
\]
\[
MS_{\text{temperature}} / MS_{\text{within}} \sim F(df_{\text{temperature}}, df_{\text{within}})
\]
\[
MS_{\text{interaction}} / MS_{\text{within}} \sim F(df_{\text{interaction}}, df_{\text{within}})
\]

Two Way ANOVA in R

```r
> wash=scan()
1: 4 5 6 5 7 9 8 12 10 12 11 9
13: 6 6 4 4 13 15 12 12
> water
[1] 1 1 1 1 2 2 2 2 3 3 3 3 1 1 1 1 2 2 2 2 2 2 3 3 3
21: 12 13 10 13
> deter=factor(c(rep(1,12),rep(2,12)))
Levels: 1 2
25:
> mean(wash)
[1] 9.083333
> water=factor(rep(gl(3,4),2))
[1] 1 1 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2 2 2 2
> deter=factor(c(rep(1,12),rep(2,12)))
Levels: 1 2
> water=factor(rep(gl(3,4),2))
```
> tapply(wash, water, mean)
  1  2  3
  5.00 11.00 11.25
> tapply(wash, deter, mean)
  1  2
  8.16667 10.00000
> tapply(wash, deter:water, mean)
  1:1 1:2 1:3 2:1 2:2 2:3
  5.0 9.0 10.5 5.0 13.0 12.0
> anova(lm.deter)
Analysis of Variance Table

Response: wash

                     Df Sum Sq Mean Sq F value    Pr(>F)
  deter              1 20.167  20.167 9.8108 0.005758 **
  water             2 200.333 100.167 48.7297 5.44e-08 ***
  deter:water       2 16.333  8.167  3.9730 0.037224 *
  Residuals         18 37.000  2.056
---
Signif. codes:  < 0.001 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

> matrix(round(fitted(lm.deter),1),byrow=T,nrow=2)
 [1,]  5  5  5  5  9  9  9  9 10.5 10.5 10.5 10.5
 [2,]  5  5  5  5 13 13 13 13 12.0 12.0 12.0 12.0
> matrix(round(residuals(lm.deter),1),byrow=T,nrow=2)
 [1,] -1  0  1  0 -2  0 -1  3 -0.5  1.5  0.5 -1.5
 [2,]  1  1 -1 -1  0  2 -1 -1  0.0  1.0 -2.0  1.0
> matrix(round(residuals(lm.deter),1)^2,byrow=T,nrow=2)
 [1,]  1  0  1  0  4  0  1  9  0.25  2.25  0.25  2.25
 [2,]  1  1  1  1  0  4  1  1 10.00  1.00  4.00  1.00
interaction.plot(water, deter, wash)