Gravitational allocation to Poisson points

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joint work with

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Let $\Xi$ be a discrete subset of $\mathbb{R}^d$.

An allocation (of Lebesgue measure to $\Xi$) is a measurable function $\psi : \mathbb{R}^d \to \Xi \cup \{\infty\}$ that satisfies

\[
\text{Vol}(\psi^{-1}(\infty)) = 0, \\
\text{Vol}(\psi^{-1}(z)) = 1, \quad z \in \Xi,
\]

where $\text{Vol}(\cdot)$ is Lebesgue measure in $\mathbb{R}^d$.

For $x \in \Xi$, we call $\psi^{-1}(z)$ the cell allocated to $z$. 
Examples

Figure: (a) The two-dimensional stable marriage allocation for a Poisson process (picture due to Alexander E. Holroyd). (b) The gradient flow allocation (picture due to Manjunath Krishnapur).
Let $Z$ be a translation-invariant simple point process in $\mathbb{R}^d$ with unit intensity.

An **invariant allocation rule** (of Lebesgue measure to $Z$) is a measurable mapping $Z \rightarrow \psi_Z$ such that:

1. a.s. $\psi_Z$ is an allocation of Lebesgue measure to $Z$, and
2. the mapping $Z \rightarrow \psi_Z$ is translation-equivariant, i.e.

$$\psi_{Z+x}(y+x) \equiv \psi_Z(y) + x.$$

If a.s. all the cells are bounded, one can consider the **allocation diameter**

$$X = \text{diam}(\psi_Z^{-1}(\psi_Z(0))).$$

One object of interest: The rate of decay of the tail $\mathbb{P}(X > R)$. 

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Talagrand: Randomized allocations for \( d \geq 3 \). Holroyd & Peres used this to construct an allocation for the Poisson process with 
\[
\mathbb{P}(\psi_Z(0) > R) \leq C \exp(-cR^d),
\]
where \( \psi_Z(0) \) is the typical allocation distance.

Related results on matchings between random point configurations in a finite setting by Ajtai, Komlós, and Tusnády ('84), Leighton and Shor ('89), and Talagrand ('94).

Holroyd & Peres: If \( d = 1, 2 \) and \( Z \) is a standard Poisson point process of unit intensity in \( \mathbb{R}^d \), then the allocation diameter \( X \) of any invariant rule satisfies 
\[
\mathbb{E}X^{d/2} = \infty.
\]

Hoffman, Holroyd & Peres: For arbitrary translation-invariant point process in \( \mathbb{R}^d, d \geq 1 \), constructed the stable marriage allocation.

In the stable marriage allocation, a.s.

1. all the cells are bounded and contain their owners,
2. but not all are connected,
3. and when \( Z \) is a Poisson point process the allocation diameter \( X \) satisfies 
   \[
   \mathbb{E}X^d = \infty.
   \]
Figure: The 2-diml. stable marriage allocation for a Poisson process

Construction: Each star (point of the process) grows a ball at unit rate and captures all the sites it reaches first, until it is sated (has obtained volume 1).
Allocation to the zeros of the Gaussian Entire Function (GEF)

\[ f(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}, \quad z \in \mathbb{C}, \]

where \((\xi_n)_{n=1}^{\infty}\) are i.i.d. standard complex Gaussian random variables.

Cell of each zero \(z\) is defined as the \textbf{basin of attraction} of \(z\) with respect to the flow induced by the random planar vector field \(F(z) = z - (\nabla \log |f|)(z)\). (See next slide for a picture of the corresponding potential \(U(z) = \log |f|(z) - \frac{1}{2} |z|^2\).)

Construction due to Nazarov, Sodin & Volberg based on an idea suggested by Tsirelson.
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**Figure:** The random planar potential (courtesy of Manjunath Krishnapur).
Nazarov, Sodin & Volberg showed that the cells are connected, a.s. bounded, and there exist absolute constants $C, c > 0$ such that the allocation diameter $X$ satisfies

$$ce^{-CR(\log R)^{3/2}} \leq \mathbb{P}(X \geq R) \leq Ce^{-cR(\log R)^{3/2}}, \quad R > 1.$$
Why equal area in each basin?

- Take a basin of attraction $B(z)$, and a point $x \in \partial B(z)$.
- If $\mathbf{n}$ is the outward-pointing normal vector at $x$, then by the definition of the basin of attraction, $F(x) \cdot \mathbf{n} = 0$.
- Thus, the oriented surface integral
  \[
  \int_{\partial B(z)} F(x) \cdot \mathbf{n} \, dS = 0.
  \]
- Now
  \[
  \text{div}(F) = 2 - 2\pi \sum_{i=1}^{\infty} \delta_{z_i},
  \]
  where $(z_i)_i$ is the set of zeros.
- Thus, by the divergence theorem,
  \[
  \int_{\partial B(z)} F(x) \cdot \mathbf{n} \, dS = \int_{B(z)} \text{div}(F) \, dx = 2\text{Vol}(B(z)) - 2\pi.
  \]
- Combining, we get $\text{Vol}(B(z)) = \pi$. 

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Let $\mathcal{Z}$ be a standard Poisson process in $\mathbb{R}^d$.

Consider the random vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$F(x) = \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

where the summands are arranged in order of increasing distance from $x$.

First investigated in work of S. Chandrasekhar. Later work by Heath & Shepp.
Convergence of the force at the origin

- Since $\mathcal{Z} = (z_i)_i$ is a Poisson process in $\mathbb{R}^d$ with unit intensity, it follows that $(|z_i|^d)_i$ is a Poisson process in $\mathbb{R}$ with intensity $\kappa_d$, where $\kappa_d$ is the volume of the unit ball in $\mathbb{R}^d$.

- Thus, we have

$$\frac{|Z_i|}{i^{1/d}} \xrightarrow{i \to \infty} \kappa_d^{-1/d} \quad \text{a.s.}$$

- If we condition on $(|z_i|)_i$, then each $z_i$ is distributed uniformly on the sphere of radius $|z_i|$ around the origin, and they are independent.

- For each $i$, $z_i/|z_i|^d$ has conditional mean 0 and variance bounded by $O(|z_i|^{-2(d-1)}) = O(i^{-2(d-1)/d})$.

- Thus, a.s. convergence of $F(0)$ follows from the Kolmogorov 3-series theorem. By stationarity, same is true for every $F(x)$.

- In our paper we have shown that a.s. the series converges everywhere on $\mathbb{R}^d \setminus \mathcal{Z}$ to give a translation-invariant continuously differentiable random function.
Each component of the vector $F(0)$ has a stable distribution whose parameter can be explicitly computed (special case by Chandrasekhar, general proof by Heath & Shepp). We give a sketch below.

If $F_1, \ldots, F_n$ are i.i.d. copies of the force $F$, then their sum is the force exerted by the union of $n$ copies of the poisson process, which is Poisson scaled by $n^{-1/d}$.

Thus, $F_1(0) + \cdots + F_n(0)$ has same law as $n^{(d-1)/d} F(0)$.

It follows that $F(0)$ is $d/(d - 1)$ stable.
A rearrangement identity

Lemma
For each \( u, x \in \mathbb{R}^d \), let

\[
G^\{u\}(x) = \sum_{\|z_i - u\| \uparrow} \frac{z_i - x}{|z_i - x|^d}.
\]

Then for any \( x, u, v \in \mathbb{R}^d \) we have \( G^\{u\}(x) - G^\{v\}(x) = \kappa_d (u - v) \) a.s., where \( \kappa_d \) is the volume of the unit ball in \( \mathbb{R}^d \).

Proof:

- If \( N_{u,x} \) is the number of stars in the ball \( B(u, |u - x|) \), then

\[
\mathbb{E} \left[ G^\{u\}(x) \mid N_{u,x} \right] = N_{u,x} \cdot \frac{u - x}{|u - x|^d}.
\]

Thus, \( \mathbb{E}(G^\{u\}(x) - G^\{v\}(x)) = \kappa_d (u - v) \).

Contd. on next slide...
Now, let $R > 0$ be large, and consider the truncated series

$$G_R^{\{u\}}(x) = \sum_{|z_i - u| < R} \frac{z_i - x}{|z_i - x|^d}.$$ 

We show that $\text{Var}(G_R^{\{u\}}(x) - G_R^{\{v\}}(x)) \to 0$ as $R \to \infty$. This suffices to complete the proof of the rearrangement identity.

*Contd. on next slide...*
Since $G_R^{\{u\}}(x) - G_R^{\{v\}}(x)$ is the sum of independent contributions from the $E_j$'s, the variance can be easily bounded.
Consider now the integral curves $\Gamma(t)$ of the vector field $F$, that is, solutions of the equation

$$\dot{\Gamma}(t) = F(\Gamma(t)).$$

We call these curves the \textbf{gravitational flow curves}.

Denote by $\Gamma_x$ the integral curve with initial condition $\Gamma_x(0) = x$.

To each center $z \in \mathcal{Z}$, define its \textbf{basin of attraction}

$$B(z) = \{x \in \mathbb{R}^d \setminus \mathcal{Z} \mid \Gamma_x(t) \text{ ends at } z\} \cup \{z\}.$$ 

Define the \textbf{gravitational allocation rule}

$$\psi_{\mathcal{Z}}(x) = \begin{cases} 
  z & x \in B(z) \text{ for } z \in \mathcal{Z}, \\
  \infty & x \not\in \bigcup_{z \in \mathcal{Z}} B(z).
\end{cases}$$
Main result

Theorem

The mapping $\mathcal{Z} \rightarrow \psi_{\mathcal{Z}}$ is an allocation rule of Lebesgue measure to the Poisson point process $\mathcal{Z}$. Almost surely all the cells $\psi^{-1}(z)$ are bounded. The allocation diameter $X = \text{diam}(\psi^{-1}(\psi(0)))$ satisfies the following tail bounds: In dimensions 4 and higher, we have

$$\mathbb{P}(X > R) \leq C_1 \exp \left[ - c_2 R (\log R)^{\frac{d-2}{d}} \right]$$

for some constants $C_1, c_2 > 0$ (depending on the dimension $d$) and all positive $R$. In dimension 3, for any $\alpha > 0$ there exist constants $C_1, c_2 > 0$ (depending on $\alpha$) such that for all $R > 0$ we have

$$\mathbb{P}(X > R) \leq C_1 \exp \left[ - c_2 \frac{R}{(\log R)^{\frac{4}{3} + \alpha}} \right].$$

(2)
Figure: Simulation of a cell in 3-dimensional gravitational allocation
We closely follow a technique introduced in an earlier version of the paper of Nazarov, Sodin & Volberg, but several new ideas are required to carry out the steps.

For $L > 0$ and $x \in \mathbb{R}^d$ denote by $Q(x, L)$ the box $x + [-L, L]^d$.

Let $E_R$ denote the event that there exists an integral curve $\Gamma(t)$ connecting $\partial Q(0, R)$ and $\partial Q(0, 2R)$.

Easy to see: If $X$ is the diameter of the basin containing 0, then for all $R > 0$,

$$\mathbb{P}(X \geq R) \leq 2\mathbb{P}(E_{cR})$$

for some constant $c$ depending only on the dimension.
Introduce a gravitational potential energy function $U(x)$ whose differences $U(x) - U(y)$ are the line integrals of the gravitational force.

$$U(x) = \frac{1}{d - 2} \lim_{T \to \infty} \left[ \sum_{i : |z_i - x| < T} \frac{-1}{|z_i - x|^{d-2}} + \frac{d^{d-1}}{2} T^2 \right].$$

Converges in $d \geq 5$.

Most important observation: Since $F(x) = -\nabla U(x)$, therefore

$$\frac{d}{dt} U(\Gamma(t)) = \langle \dot{\Gamma}(t), \nabla U(\Gamma(t)) \rangle$$

$$= \langle F(\Gamma(t)), -F(\Gamma(t)) \rangle = -|F(\Gamma(t))|^2.$$

In particular, the potential always decreases along a gradient flow.
We bound $\mathbb{P}(E_R)$ in terms of discrete events, by dividing space into a grid $S$ of cubes of side length $r \approx (\log R)^{2/d}$.

We fix $B = R^{8/9}$ and say that a cube is **bad** if $|U(x) - U(y)| \leq Br/R$ for all $x, y$ in the cube, or $U(x) < -B$ for some $x$ in the cube.

In other words, a cube is bad if either the gravitational force or the potential is unusually small inside the cube.

If $U(x) \leq B$ for all $x \in Q(0, 2R)$ (a high probability event), then $E_R$ happens only if there is a sequence of $cR/r$ bad cubes.
Here $T_1$ is the time at which the curve $\Gamma$ enters the phase $U(x) < -B$. 
Final steps

- Step 1: Show that the probability of a cube being bad is small, by further subdividing into smaller cubes and applying Markov’s inequality to get a crude bound.

- Step 2: Get an exponentially small bound on the joint probability of a collection of well-separated cubes being bad (the percolation step).

- Step 3: Bound $\mathbb{P}(E_R)$ by summing over all connected sequences of cubes.

- The percolation step turns out to be a lot of work, mainly due to the heavy-tailed nature of the force (unlike the Nazarov-Sodin-Volberg scenario, where the dependence decreases exponentially with distance).
Key components of the percolation step

In lay terms, we have to show that...

(i) The force $F$ cannot be too small at a large number of well-separated points, and

(ii) The potential $U$ cannot be too large (negatively) at a large number of well-separated points.

To carry out the program, we first need some control at single points, and then execute dependent percolation arguments.
The \((a, r)\)-truncated potential difference at \(x\) is defined as
\[
U^a_r(x) = \sum_{a < |z_i| < r, |z_i - x| > r} (|z_i|^{2-d} - |z_i - x|^{2-d}).
\]

**Proposition**

*There exists two constants \(C_0\) and \(C_1\) depending only on \(d\), so that*

\[
\mathbb{P}(|U^a_r - \mathbb{E}(U^a_r)| \geq t) \leq 2 \exp(-C_0 r^{d-2} t \log(t/C_1 r^2))
\]

*for all \(t\) above a threshold that depends on \(d\), \(r\), and \(|x|\).*

1. **For** \(d \geq 5\), **we only need that** \(t \geq C_1 r^2\) (**the threshold does not depend on** \(|x|\)).

2. **For** \(d = 4\), **we need** \(t \geq C_1 r^2\), **and also**
   \(t \geq C_1 r^2 \log(t/C_1 r^2) \log(|x|/r)\).

3. **For** \(d = 3\), **we need** \(t \geq C_1 r^2\), **and** \(t \geq C_1 |x| r \log(t/C_1 r^2)\), **and**
   \(t \geq C_1 r^2 (\log(t/C_1 r^2))^2 \log(|x|/r)\).

*For** \(d = 3\), **if** \(1 \leq r \leq |x|\),
\[
\mathbb{P}(|U^a_r - \mathbb{E}(U^a_r)| \geq t) \leq 2 \exp(-t^2/2|x|).
\]
To control $U$ simultaneously at a large number of points, we need similar tail bounds for $\nabla U = F$.

The proof is complicated by the fact that the individual summands are heavy-tailed.

The estimates are harder to obtain in $d = 3, 4$ than $d \geq 5$ because of the dependence on $x$. 
The main step is the following bound on the joint density of the forces at multiple points.

**Lemma**

*Suppose we have* \( x_1, \ldots, x_N \in \mathbb{R}^d \) *with* \( |x_i - x_j| > S \) *for every* \( i \neq j \). *Fix* \( \lambda > 0 \), *and let*

\[
E = \left\{ \text{There is at least one star in } B(x_i, \lambda) \text{ for every } 1 \leq i \leq N \right\}.
\]

*Let* \( \mathcal{M} \) *the* \( \sigma \)-algebra generated by the stars in* \( \left( \bigcup_{i=1}^{N} B(x_i, S) \right)^c \). *Then there exist constants* \( c_0, C_1 > 0 \) *such that if* \( \lambda < c_0 S (\log N)^{-1/d} \), *then conditioned on the event* \( E \) *and on the* \( \sigma \)-algebra \( \mathcal{M} \), *almost surely the joint density of* \( (F(x_i))_{1 \leq i \leq N} \) *exists and is bounded from above by* \( (C_1 \lambda^{d^2-d})^N \).
Proof of the lemma involves actual computation and bounding of the inverse Jacobian.

The bound is probably suboptimal, but suffices for our purposes.

Paper is available on arXiv:
http://arxiv.org/abs/math.PR/0611886