Superconcentration

Sourav Chatterjee

(UC Berkeley)
This talk is about three properties of Gaussian fields:

1. Multiple valleys
2. Chaos
3. Superconcentration

Based on two papers:

1. "Chaos, concentration, and multiple valleys." arXiv:0810.4221v2 [The three properties are defined here, and shown to be equivalent, along with a number of examples.]
2. "Disorder chaos and multiple valleys in spin glasses." arXiv:0907.3381v1 [We show that the Sherrington-Kirkpatrick model has all three properties.]

Examples: directed polymers, last passage percolation, spin glasses, the discrete Gaussian free field, random matrix eigenvectors, fitness models of evolutionary biology, etc.

We will illustrate the theory through a single example in this talk: the Sherrington-Kirkpatrick model of spin glasses.

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The Sherrington-Kirkpatrick model

- System of $N$ particles, each with spin $+1$ or $-1$. State space: $\{-1, 1\}^N$. 

Spin glass: particles have magnetic interactions with each other; both ferromagnetic and anti-ferromagnetic interactions present in the same material.

Let $(g_{ij})_{1 \leq i, j \leq N}$ be i.i.d. standard Gaussian random variables.

Sherrington-Kirkpatrick model: For each state $\sigma \in \{-1, 1\}^N$, define the energy of $\sigma$ as

$$H_N(\sigma) := -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j.$$ 

$H_N$ is the Hamiltonian (or energy function) in the Sherrington-Kirkpatrick model of spin glasses. Note that it is a random function on $\{-1, 1\}^N$. 

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Some history

- Introduced by Sherrington and Kirkpatrick in 1975; first breakthrough by Thouless-Anderson-Palmer in 1977; revolutionary development of the broken replica method by Parisi and Mézard in late 70’s and early 80’s; first rigorous analysis of the high temperature phase by Aizenman-Lebowitz-Ruelle in 1987, and later by Fröhlich-Zegarlinski, Comets-Neveu; high temperature phase under nonzero external field studied by Talagrand and Shcherbina in the late 90’s; great advancement in the rigorous understanding of the low temperature phase due to breakthroughs of Guerra, Toninelli, Talagrand and Panchenko between 2001 and 2008. Most significant breakthrough: Proof of the Parisi formula by Talagrand in 2003.

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For $\sigma^1, \sigma^2 \in \{-1, 1\}^N$, define

$$R_{\sigma^1, \sigma^2} = R_{1,2} := \frac{\sum_{i=1}^N \sigma^1_i \sigma^2_i}{N}.$$
The overlap

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$$R_{\sigma^1, \sigma^2} = R_{1,2} := \frac{\sum_{i=1}^{N} \sigma_i^1 \sigma_i^2}{N}.$$ 

In the spin glass literature, the quantity $R_{1,2}$ is called the overlap between the configurations $\sigma^1$ and $\sigma^2$. 
Recall: The energy of a state $\sigma$ is defined as

$$H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j,$$

where $g_{ij}$ are i.i.d. standard Gaussian r.v.
The multiple valley question

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Is it true that with high probability, there exists a large number of mutually nearly-orthogonal states that all have nearly minimum-energy?


No rigorous formulation or results till now.
A counterexample

To realize the non-triviality of the question, consider a slightly different Gaussian field $Y_N$ on $\{-1,1\}^N$, defined as

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Clearly, $Y_N$ is minimized at $\hat{\sigma}$, where $\hat{\sigma}_i = \text{sign}(g_i)$. 

Thus, the field $Y_N$ does not have multiple valleys.

This is true in spite of $Y_N(\sigma)$ and $Y_N(\sigma')$ being nearly independent for most $\sigma, \sigma'$. 

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Since

$$Y_N(\sigma) = -\sum_{i: \sigma_i = \hat{\sigma}_i} |g_i| + \sum_{i: \sigma_i \neq \hat{\sigma}_i} |g_i|,$$

we see that if $\sigma$ is another configuration that is near-minimal for $Y_N$, then $\sigma$ must agree with $\hat{\sigma}$ at nearly all coordinates.
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Weak resolution of the multiple valley conjecture

Recall: \( R_{\sigma^1, \sigma^2} = \frac{1}{N} \sum \sigma^1_i \sigma^2_i \), \( H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum g_{ij} \sigma_i \sigma_j \).
Weak resolution of the multiple valley conjecture

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Theorem (C. '09)

There are constants \( r_N \to \infty, \gamma_N \to 0, \epsilon_N \to 0, \text{ and } \delta_N \to 0 \) such that with probability at least \( 1 - \gamma_N \), there is a set \( A \subseteq \{-1, 1\}^N \) satisfying

(a) \( |A| \geq r_N \),

(b) \( R_{\sigma_1, \sigma_2}^2 \leq \epsilon_N \) for all \( \sigma_1, \sigma_2 \in A, \sigma_1 \neq \sigma_2 \), and

(c) For all \( \sigma \in A \),

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\left| \frac{H_N(\sigma)}{\min_{\sigma' \in \{-1,1\}^N} H_N(\sigma')} - 1 \right| \leq \delta_N.
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Quantitatively, we can take \( r_N = (\log N)^{1/8} \), \( \delta_N = (\log N)^{-1/8} \), \( \epsilon_N = e^{-(\log N)^{1/8}} \) and \( \gamma_N = C(\log N)^{-1/12} \), where \( C \) is an absolute constant. However these are not necessarily the best choices.
The S-K model at inverse temperature $\beta \geq 0$ defines a probability measure $G_N$ on $\{-1, 1\}^N$ through the formula

$$G_N(\{\sigma\}) := Z(\beta)^{-1} e^{-\beta H_N(\sigma)},$$

(1)

where $Z(\beta)$ is the normalizing constant. The measure $G_N$ is called the Gibbs measure. Recall that

$$H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j.$$
Disorder chaos in the S-K model

Suppose $\sigma^1$ and $\sigma^2$ are two configurations drawn independently from the Gibbs measure $G_N$. Recall:

$$R_{1,2} = \frac{1}{N} \sum \sigma_i^1 \sigma_i^2.$$
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- This is not true when $\beta > 1$. In this regime, $R_{1,2}$ has a non-degenerate limiting distribution (the Parisi measure).

Not to be confused with temperature chaos.

Again, no rigorous formulation or proof in the past. Seems related to noise-sensitivity, although we do not understand the exact connection.
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- Suppose we choose $\sigma^2$ from a new Gibbs measure $G'_N$, based on a new Hamiltonian $H'_N$ obtained by applying a small perturbation to $H_N$. 

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We consider two kinds of perturbation.

Discrete perturbation:

Replace a randomly chosen fraction $p$ of the couplings ($g_{ij}$) by independent copies. The resulting Gibbs measure will be called the $p$-perturbed measure.

Theorem (C. '09)

Let $\sigma_1$ be chosen from the original Gibbs measure and $\sigma_2$ is chosen from the $p$-perturbed measure. Then

$$E(R_1^2, R_2^2) \leq C \beta p \log N,$$

where $C$ is an absolute constant and the expectation is taken over all randomness.
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$$\mathbb{E}(R_{1,2}^2) \leq \frac{C \beta}{p \log N},$$

where $C$ is an absolute constant and the expectation is taken over all randomness.
How to prove multiple valleys using chaos

- Choose $\sigma^1$ from the Gibbs measure $G_N$ at inverse temperature $\beta$ and $\sigma^2$ from the $p$-perturbed measure $G'_N$. 

Suppose $\beta = \beta(N) \to \infty$ and $p = p(N) \to 0$ sufficiently slowly so that chaos holds.

Then due to chaos, $\sigma^1$ and $\sigma^2$ are approximately orthogonal.

Since $\beta \to \infty$, $\sigma^1$ nearly minimizes $H_N$ and $\sigma^2$ nearly minimizes $H'_N$.

But, since $p \to 0$, $H_N \approx H'_N$.

Thus, $\sigma^1$ and $\sigma^2$ both nearly minimize $H_N$.

This procedure finds two states that have nearly minimal energy and are nearly orthogonal. Repeating this procedure, we find many such states.
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- Then due to chaos, $\sigma^1$ and $\sigma^2$ are approximately orthogonal.

Since $\beta \to \infty$, $\sigma^1$ nearly minimizes $H_N$ and $\sigma^2$ nearly minimizes $H'_N$.
- But, since $p \to 0$, $H_N \approx H'_N$.
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Sourav Chatterjee  
Superconcentration
Choose $\sigma^1$ from the Gibbs measure $G_N$ at inverse temperature $\beta$ and $\sigma^2$ from the $p$-perturbed measure $G'_N$.

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How to prove multiple valleys using chaos

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The free energy of the S-K model at inverse temperature $\beta$ is:

$$F_N(\beta) := \frac{1}{\beta} \log \sum_{\sigma} e^{-\beta H_N(\sigma)}.$$ 

Recall: $H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum g_{ij} \sigma_i \sigma_j$. 

It is known that $\text{Var}(F_N(\beta)) \leq \text{const}$ if $\beta < 1$. 

However, $\text{Var}(F_N(\beta)) \leq N$ was the best available bound for $\beta > 1$ till now. 

We claim that for any $\beta$, $F_N(\beta)$ is superconcentrated, meaning that $\text{Var}(F_N(\beta)) = o(N)$. 

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Sourav Chatterjee
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Theorem (C. ’09)

Let $F_N(\beta)$ be the free energy of the S-K model. Then for any $\beta$, 

$$\text{Var}(F_N(\beta)) \leq \frac{CN \log(1 + C\beta)}{\log N},$$

where $C$ is an absolute constant.
Superconcentration of the free energy

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- Has been called ‘sublinear variance’ or ‘submean variance’ before. ‘Superconcentration’ is probably more evocative.
- In [Chatterjee ’08], it was shown that superconcentration is equivalent to chaos and multiple valleys in a general setting. Therefore, it is more than just a curious phenomenon.
The following theorem shows that the system is chaotic if 
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Suppose $\sigma^1$ is drawn from the Gibbs measure and $\sigma^2$ from the $p$-perturbed Gibbs measure. Then

$$\mathbb{E}(R_{1,2}^2) \leq C \frac{\text{Var}(F_N(\beta))}{pN} + C\beta N^{-1/2},$$

where $C$ is an absolute constant.
Continuous perturbation

Replace $g_{ij}$ by $ag_{ij} + bg_{ij}'$, where $(g_{ij}')$ is another set of independent standard Gaussian random variables and $a^2 + b^2 = 1$. When $a \approx 1$, we say that the perturbation is small. A convenient way to parametrize the perturbation is to set $a = e^{-t}$, where $t$ is a parameter that we call 'time'. Perturbing the couplings up to time $t$ corresponds to running an Ornstein-Uhlenbeck flow at each coupling for time $t$, with initial value $g_{ij}$. The new Gibbs measure will be called the $t$-perturbed measure.
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Theorem (C. ’09)

Let $\sigma^1$ be chosen from the original Gibbs measure and $\sigma^2$ be chosen from the $t$-perturbed measure. Then there is an absolute constant $C$ such that for any positive integer $k$,

$$\mathbb{E}(R_{1,2}^{2k}) \leq (Ck)^k N^{-k \min\{1, \frac{t}{C \log(1+C\beta)}\}}.$$
How to prove superconcentration ⇐⇒ continuous chaos

Theorem (C. ’09)

Let $\phi(t)$ denote $\mathbb{E}(R^2_{1,2})$ when $\sigma^2$ is drawn from the $t$-perturbed measure. Let $F_N(\beta)$ be the free energy. Then

$$\text{Var}(F_N(\beta)) = N \int_0^\infty e^{-t} \phi(t) dt.$$
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- “$\text{Var}(F_N(\beta)) = o(N)$ if $\phi(t) = o(1)$ for all $t > o(1)$.”
- We will also show that $\phi$ is a nonnegative and decreasing function. This proves the converse implication.
How to prove superconcentration \iff continuous chaos

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- “$\text{Var}(F_N(\beta)) = o(N)$ if $\phi(t) = o(1)$ for all $t > o(1)$.”
- We will also show that $\phi$ is a nonnegative and decreasing function. This proves the converse implication.
- By our chaos theorem for continuous perturbation, $\phi(t) \leq CN^{-\min\{1,t/C(\beta)\}}$. This shows that

$$\text{Var}(F_N(\beta)) \leq \frac{C(\beta)N}{\log N}.$$
What have we sketched till now?

- Continuous chaos $\iff$ superconcentration of free energy
  $\implies$ discrete chaos $\implies$ multiple valleys.

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Sourav Chatterjee  Superconcentration
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Suppose $\sigma^1$ is drawn from the Gibbs measure and the $\sigma^2$ from the $t$-perturbed measure. Recall: $R_{1,2} = \frac{1}{N} \sum \sigma_1^1 \sigma_2^1$. Let

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$$= \mathbb{E} \left( \sum_{\sigma^1, \sigma^2} \left( \frac{\sum \sigma^1_i \sigma^2_i}{N} \right)^{2k} e^{\frac{\beta}{\sqrt{N}} \sum_{i<j}(g_{ij}\sigma^1_i \sigma^1_j + g^t_{ij}\sigma^2_i \sigma^2_j)} \right) \frac{\sum_{\sigma^1, \sigma^2} e^{\frac{\beta}{\sqrt{N}} \sum_{i<j}(g_{ij}\sigma^1_i \sigma^1_j + g^t_{ij}\sigma^2_i \sigma^2_j)}}{\sum_{\sigma^1, \sigma^2} e^{\frac{\beta}{\sqrt{N}} \sum_{i<j}(g_{ij}\sigma^1_i \sigma^1_j + g^t_{ij}\sigma^2_i \sigma^2_j)}}$$

where $g^t_{ij} := e^{-t}g_{ij} + \sqrt{1 - e^{-2t}}g'_{ij}$. 

To show: For all $t$, $\phi_k(t) \leq CN^{-k \min\{1, t/C\}}$ for some constant $C$ depending on $\beta$. 

By repeated applications of differentiation and Gaussian integration-by-parts, we show that $(-1)^j \phi^{(j)}(t) \leq 0$ for all $t$ and $j$. Here $\phi^{(j)}(t)$ denotes the $j$th derivative of $\phi(t)$. Such functions are called completely monotone.
Proof of chaos under continuous perturbation - 1

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Sourav Chatterjee

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To show: For all $t$, $\phi_k(t) \leq C N^{-k \min\{1, t/C\}}$ for some constant $C$ depending on $\beta$.

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Such functions are called completely monotone.
By a classical theorem of Bernstein about completely monotone functions, there is a probability measure $\mu_k$ on $[0, \infty)$ such that

$$\phi_k(t) = \phi_k(0) \int_0^\infty e^{-xt} d\mu_k(x).$$

By Hölder's inequality and the above representation, it follows that for $0 \leq t < s$,

$$\phi_k(t) \leq \phi_k(0) \frac{1 - t/s}{\phi_k(s) t/s}.$$

In other words, chaos under large perturbations implies chaos under small perturbations.

Thus, it suffices to prove that $\phi_k(s) \leq \text{const} \cdot N^{-k}$ for sufficiently large $s$. 

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Proof of chaos under continuous perturbation - 2

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Sourav Chatterjee  Superconcentration
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Using various tricks, one can show that for any $k$ and $s$, 

$$\phi_k'(s) \geq -2 N \beta^2 e^{-s \phi_k(s) + 1}.$$ 

Thus, we have a chain of differential inequalities. 

It is possible to manipulate this chain to conclude that 

$$\phi_k(s) \leq 2^{-2 N \sum \sigma_1, \sigma_2 (\sigma_1 \cdot \sigma_2) / N^2 \exp(2 \beta^2 e^{-s (\sigma_1 \cdot \sigma_2) / N})}.$$ 

The right hand side is bounded by $const. N^{-k}$ iff $s$ is sufficiently large. (Related to the fact that when $Z \sim N(0, 1)$, $E(e^{\alpha Z^2}) < \infty$ iff $\alpha < 1/2$.) This completes the proof.
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Proof of chaos under continuous perturbation - 3

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Other results

- Presence of chaos, superconcentration and multiple valleys in **directed polymers** in a Gaussian random environment [C. ’08].

- Implication in directed last passage percolation: many approximately longest paths that are all approximately disjoint.

- Chaos in eigenvectors of Gaussian random matrices [C. ’08].

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