Decay of correlations in the random field Ising model

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Ising model

- Let Λ be a finite subset of $\mathbb{Z}^d$.
- Boundary: $\partial\Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda : x \text{ is adjacent to some } y \in \Lambda\}$.
- Configurations: $\Sigma = \{-1, 1\}^\Lambda$
- Boundary conditions: $\Gamma = \{-1, 1\}^{\partial\Lambda}$.
- External field: $\Phi = \mathbb{R}^\Lambda$.
- For $\sigma \in \Sigma$, $\gamma \in \Gamma$ and $\phi \in \Phi$, define the energy of σ as

$$H_{\gamma,\phi}(\sigma) := -\frac{1}{2} \sum_{\substack{x,y\in\Lambda, \\ x\sim y}} \sigma_x \sigma_y - \sum_{\substack{x\in\Lambda,\, y\in\partial\Lambda, \\ x\sim y}} \sigma_x \gamma_y - \sum_{x\in\Lambda} \phi_x \sigma_x,$$

  where $x \sim y$ means that x and y are neighbors.
- Inverse temperature: $\beta \in [0, \infty]$.
- The Ising model on Λ with boundary condition γ, inverse temperature β, and external field ϕ, is the probability measure on Σ that puts mass proportional to $e^{-\beta H_{\gamma,\phi}(\sigma)}$ at σ.
- When $\beta = \infty$, it is the uniform probability measure on configurations that minimize the energy (ground states).

Random field Ising model

- Now suppose that the external field $\phi = (\phi_x)_{x \in \Lambda}$ consists of i.i.d. random variables instead of fixed constants.
- Then the probability measure defined above becomes a random probability measure.
- This is known as the random field Ising model (RFIM).
- We will refer to the law of ϕ_x as the random field distribution.

History

- The RFIM was introduced by Imry and Ma in 1975 as a simple example of a disordered system.
- Imry and Ma predicted that the model does not have an ordered phase in dimensions one and two, but does exhibit a phase transition in dimensions three and higher.
- A competing conjecture was put forward by Parisi and Sourlas, who argued that the phase transition appears in dimension four, not three.
- The competing conjectures led to spirited debate in the physics community, and the question remained unresolved for many years.
- Finally, in a rare instance of mathematicians settling a debate for physicists, Bricmont and Kupiainen (1988) proved that indeed there is an ordered phase in $d \geq 3$ and Aizenman and Wehr (1990) proved that there is no ordered phase in $d \leq 2$.

Decay of correlations

- Recall: RFIM defined on $\Lambda \subseteq \mathbb{Z}^d$. $\Gamma =$ set of boundary conditions.
- Fix some β, and let $\langle \sigma_x \rangle_\gamma$ denote the expected value of σ_x at inverse temperature β and boundary condition γ.
- We say that correlations decay if

$$\sup_{\gamma, \gamma' \in \Gamma} |\langle \sigma_x \rangle_\gamma - \langle \sigma_x \rangle_{\gamma'}| \to 0$$

  in probability as $\Lambda \uparrow \mathbb{Z}^d$, with x and β remaining fixed.
- In other words, the effect of the boundary condition on the law of the spin at some interior point becomes negligible as the distance of the point from the boundary becomes large.
- The Bricmont–Kupiainen theorem says that this does not happen for large β when $d \geq 3$, and the Aizenman–Wehr theorem says that this happens for all β when $d \leq 2$.

Rate of decay

- It is easy to show that the 1D model has exponential decay of correlations at all β.
- There is a conjecture that the same is true in 2D.
- Unfortunately, the Aizenman–Wehr theorem uses ergodic theory in a crucial way, making it unclear how to extract any quantitative information.
- The following theorem gives the first progress on this question.

Theorem (C., 2017)

Let the random field distribution be $\mathcal{N}(0, v)$. Take any $x \in \Lambda$ and let n be the distance of x from $\partial\Lambda$. Then

$$\mathbb{E}\left(\sup_{\gamma,\gamma' \in \Gamma} |\langle \sigma_x \rangle_\gamma - \langle \sigma_x \rangle_{\gamma'}|\right) \le \frac{C(1 + v^{-1/2})}{\sqrt{\log \log n}},$$

where C is a universal constant. In particular, the bound has no dependence on β and holds even if $\beta = \infty$.

Remarks

- The question of proving exponential decay of correlations in the RFIM is one of many such open questions in statistical mechanics.
- Examples: 2D $O(n)$ models for $n \geq 3$, discrete Gaussian models, non-abelian lattice gauge theories, Yang–Mills mass gap, etc.
- While we know how to prove exponential decay at small β, proving exponential decay at large β (when expected) is usually out of the reach of existing technology.
- The proof of the theorem introduces a new method of proving decay of correlations that may useful in other problems.

FKG property

- The RFIM is an FKG system, meaning that monotone increasing functions of spins are positively correlated with each other.
- An important consequence of the FKG property is that $\langle \sigma_x \rangle_\gamma$ is an increasing function of the boundary condition γ.
- Let $+$ denote the boundary condition of all $+1$, and let $-$ be the boundary condition of all -1.
- Then monotonicity implies that for any γ,

$$\langle \sigma \rangle_+ \geq \langle \sigma \rangle_\gamma \geq \langle \sigma \rangle_-.$$

- In particular,

$$\sup_{\gamma, \gamma'} |\langle \sigma \rangle_\gamma - \langle \sigma \rangle_{\gamma'}| = \langle \sigma \rangle_+ - \langle \sigma \rangle_-.$$

Sketch of the proof: Step 1

- For simplicity, assume that Λ is an $n \times n$ square, and that the random field distribution in $\mathcal{N}(0,1)$.
- Throughout, C will denote universal constants.
- Let
$$M_+ := \sum_{x \in \Lambda} \langle \sigma_x \rangle_+, \quad M_- := \sum_{x \in \Lambda} \langle \sigma_x \rangle_-.$$
- Turns out that it suffices to show that
$$\mathbb{E}(M_+ - M_-) \le \frac{Cn^2}{\sqrt{\log \log n}}.$$

Sketch of the proof: Step 2

- Let $m \ll n$ be a number, to be chosen later.
- Partition Λ into a collection $\mathcal{B}$ of $m \times m$ sub-squares.
- For each $B \in \mathcal{B}$, let

$$M_+(B) := \sum_{x \in B} \langle \sigma_x \rangle_+, \quad M_-(B) := \sum_{x \in B} \langle \sigma_x \rangle_-.$$

- We will shows that for most $B \in \mathcal{B}$,

$$\mathbb{E}(M_+(B)) - \mathbb{E}(M_-(B)) \le \frac{Cm^2}{\sqrt{\log \log n}}.$$

- Summing over B, this gives

$$\mathbb{E}(M_+) - \mathbb{E}(M_-) \le \frac{Cn^2}{\sqrt{\log \log n}}.$$

Sketch of the proof: Step 3

- Free energy: Logarithm of the partition function (normalizing constant).
- Let F_γ be the free energy under boundary condition γ.
- Fix an $m \times m$ square B.
- Modify the model by replacing ϕ_x by $\phi_x + h$ for all $x \in B$.
- Let $F_\gamma(h)$ be the new free energy.
- This is useful because

$$M_+(B) = \frac{1}{\beta} F'_+(0), \quad M_-(B) = \frac{1}{\beta} F'_-(0).$$

- So, we need to show that

$$\mathbb{E}(F'_+(0)) - \mathbb{E}(F'_-(0)) \le \frac{C\beta m^2}{\sqrt{\log \log n}}.$$

- We will show this by approximating $F'_\gamma(0)$ by $(F_\gamma(h) - F_\gamma(0))/h$ for some suitable small h.

Sketch of the proof: Step 4

- Now take any boundary condition γ and any h, and create a new model by decoupling the links between B and $\Lambda \setminus B$.
- Let $\tilde{F}_\gamma(h)$ be the free energy of the new model.
- Due to the decoupling, $\tilde{F}_\gamma(h)$ decomposes as a sum of contributions from inside and outside B.
- The contribution from outside B does not depend on h, and the contribution from inside B does not depend on the boundary condition.
- Thus, there is some $\alpha(h)$ depending only on h and not on γ, such that

$$\tilde{F}_\gamma(h) - \tilde{F}_\gamma(0) = \alpha(h).$$

- Not hard to show that for any h,

$$|F_\gamma(h) - \tilde{F}_\gamma(h)| \leq Cm.$$

  (We use $d = 2$ here, through the fact that $|\partial B| \leq Cm$.)
- Combining, we get $|(F_\gamma(h) - F_\gamma(0)) - \alpha(h)| \leq Cm$.

Sketch of the proof: Step 5

- Consequently,

$$\left|\frac{F_+(h) - F_+(0)}{h} - \frac{F_-(h) - F_-(0)}{h}\right| \le \frac{Cm}{h}.$$

- It only remains to get a bound for

$$\left|\mathbb{E}(F_\gamma'(0)) - \mathbb{E}\left(\frac{F_\gamma(h) - F_\gamma(0)}{h}\right)\right|$$

  in terms of m and h, and then show that m and h can be chosen to get the desired result.
- We will now do this for plus boundary condition, the argument being similar for minus boundary.

Sketch of the proof: Step 6

- By Taylor expansion,

$$\mathbb{E}\left(\frac{F_+(h) - F_+(0)}{h}\right) - \mathbb{E}(F_+'(0)) = \sum_{k=2}^{\infty} \frac{h^{k-1}}{k!}\mathbb{E}(F_+^{(k)}(0)).$$

- Not hard to see that

$$F_+^{(k)}(0) = \sum_{x_1,\ldots,x_k \in B} \frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}},$$

- Thus,

$$\sum_{k=2}^{\infty} \frac{h^{k-1}}{k!}\mathbb{E}(F_+^{(k)}(0)) = \sum_{k=2}^{\infty}\sum_{x_1,\ldots,x_k \in B} \frac{h^{k-1}}{k!}\mathbb{E}\left(\frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}}\right).$$

Sketch of the proof: Step 7

- Since F_+ is a function of standard Gaussian random variables, we can write its L^2 norm using the Fourier expansion of F_+ in the multivariate Hermite polynomial basis of Gaussian L^2 space.
- The quantities
$$\mathbb{E}\left(\frac{\partial^k F_+}{\partial \phi_{x_1} \cdots \partial \phi_{x_k}}\right),$$
as $x_1, \ldots, x_k$ range over Λ, are its Fourier coefficients.
- In particular,
$$\mathrm{Var}(F_+) = \sum_{k=1}^{\infty} \sum_{x_1,\ldots,x_k \in \Lambda} \frac{1}{k!}\left(\mathbb{E}\left(\frac{\partial^k F_+}{\partial \phi_{x_1} \cdots \partial \phi_{x_k}}\right)\right)^2.$$
- Using standard methods from concentration of measure, one can show that $\mathrm{Var}(F_+) \leq C\beta^2 n^2$.

Sketch of the proof: Step 8

- Thus,

$$\sum_{k=1}^{\infty} \sum_{x_1,\ldots,x_k \in \Lambda} \frac{1}{k!}\left(\mathbb{E}\left(\frac{\partial^k F_+}{\partial \phi_{x_1} \cdots \partial \phi_{x_k}}\right)\right)^2 \le C\beta^2 n^2.$$

- Let

$$S_j := \sum_{k=1}^{\infty} \sum_{\substack{x_1,\ldots,x_k \in \Lambda, \\ (\log n)^{j-1} < \max_{a,b} |x_a - x_b| \le (\log n)^j}} \frac{1}{k!}\left(\mathbb{E}\left(\frac{\partial^k F_+}{\partial \phi_{x_1} \cdots \partial \phi_{x_k}}\right)\right)^2.$$

- Then $\sum_{j=1}^{\infty} S_j \le \mathrm{Var}(F_+) \le C\beta^2 n^2$.
- Thus, for any k, there exists $j \le k$ such that $S_k \le C\beta^2 n^2/k$.
- In particular, there exists $j_0 \le (\log n)/(2 \log\log n)$ such that

$$S_{j_0} \le \frac{C\beta^2 n^2 \log\log n}{\log n}.$$

Sketch of the proof: Step 9

- Let $m = (\log n)^{j_0}$, so that $m \le \sqrt{n}$.
- Then

$$\sum_{k=2}^\infty \frac{h^{k-1}}{k!}\mathbb{E}(F_+^{(k)}(0)) = \sum_{k=2}^\infty \sum_{x_1,\ldots,x_k\in B} \frac{h^{k-1}}{k!}\mathbb{E}\biggl(\frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}}\biggr)$$

$$= \sum_{k=2}^\infty \sum_{\substack{x_1,\ldots,x_k\in B,\\ \max_{a,b}|x_a-x_b|\le(\log n)^{j_0-1}}} \frac{h^{k-1}}{k!}\mathbb{E}\biggl(\frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}}\biggr)$$

$$+ \sum_{k=2}^\infty \sum_{\substack{x_1,\ldots,x_k\in B,\\ (\log n)^{j_0-1}<\max_{a,b}|x_a-x_b|\le(\log n)^{j_0}}} \frac{h^{k-1}}{k!}\mathbb{E}\biggl(\frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}}\biggr).$$

- Separately apply Cauchy–Schwarz to the two parts, and apply the bound for S_{j_0} to the second part. Then average over all choices of $B \in \mathcal{B}$.

Sketch of the proof: Step 10

- This gives

$$\left|\sum_{k=2}^{\infty} \frac{h^{k-1}}{k!} \mathbb{E}(F_+^{(k)}(0))\right| \le C\beta m^2 \left(\frac{hm}{\log n} + e^{h^2 m^2} \sqrt{\frac{\log \log n}{\log n}}\right).$$

- Combining all steps, we get

$$\begin{aligned}
&\mathbb{E}(F_+(0)) - \mathbb{E}(F_-(0)) \\
&\le \frac{C\beta m}{h} + C\beta m^2 \left(\frac{hm}{\log n} + e^{h^2 m^2} \sqrt{\frac{\log \log n}{\log n}}\right).
\end{aligned}$$

- Choosing $h = \sqrt{\log \log n}/2m$, we get

$$\mathbb{E}(F_+(0)) - \mathbb{E}(F_-(0)) \le \frac{C\beta m}{\sqrt{\log \log n}},$$

completing the proof.

Open problems

- Prove exponential decay of correlations in the 2D RFIM at large β.
- Failing that, at least prove polynomial decay.
- Find a relatively simple argument for the existence of the ordered phase in the RFIM in $d \geq 3$. The only proof, due to Bricmont and Kupiainen, uses rigorous renormalization group analysis and is very complex.
- Prove anything about the decay of correlations in the 2D Edwards–Anderson spin glass model, which is Ising model but with random couplings ($\sigma_x \sigma_y$ replaced by $g_{xy} \sigma_x \sigma_y$, where g_{xy} is random).
- Find a general approach for proving exponential decay of correlations at large β, in models where it is expected.