The endpoint distribution of directed polymers

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(Joint work with Erik Bates)

Beyond the simple random walk

- "Non-simple" random walks have been a source of fascinating problems for many years.
- Examples:
  - Random walk in i.i.d. random environment (RWRE).
  - Edge-reinforced and vertex-reinforced random walks (ERRW and VRRW).
  - Directed polymers in random environment.
- Many open questions.

Localization

- One of the most interesting aspects of such walks is strong localization: The position of the walker at time n is guaranteed to belong to a region of diameter $O(1)$ with probability $\to 1$ as $n \to \infty$.
- Models where this has been proved:
  - One dimensional RWRE, by Gantert, Peres & Shi (2010), following contributions from Sinai (1982), Golosov (1984), Shi (1998) and Révész (2005).
  - ERRW in any dimension, by Angel, Crawford & Kozma (2014) and Sabot and Tarrès (2015). Earlier results for one dimension are surveyed in Pemantle (2007).
  - Directed polymers in one spatial dimension, by Comets & Nguyen (2015), for an exactly solvable model introduced by Seppäläinen (2012).
- Strong localization in RWRE and directed polymers in dimensions higher than one is out of the reach of available techniques.

This talk

- I will start with the definition of directed polymers in random environment.
- I will then state one of our main results, which proves subsequential strong localization in directed polymers in any dimension.
- Lastly, I will give a sketch of the proof, which uses a new abstract machinery with many other consequences for directed polymers — and hopefully, future applications in other models.

Directed polymers in random environment

- Defined in the physics literature by Huse & Henley (1985) and Kardar (1985), as a model of Ising interfaces.
- First rigorous study by Imbrie & Spencer (1988).
- Tremendous interest in the probability community since the seminal work of Johansson (2000). Some of the notable contributors, in alphabetical order, are: Amir, Borodin, Carmona, Comets, Corwin, Hairer, Hu, Rassoul-Agha, Remenik, Quastel, Sepäläinen, Shiga, Vargas, Yoshida, ...
- A lot is known in one spatial dimension due to exact solvability, but not so much in higher dimensions.

Definition

- Fix some dimension d. For each $n \geq 0$ and $x \in \mathbb{Z}^d$, let $\omega_{n,x}$ be a random weight. Assume that the weights are i.i.d.
- Take any n. For any path $p = (x_0, x_1, \ldots, x_n)$ in $\mathbb{Z}^d$ with $x_0 = 0$, define
$$\omega(p) := \sum_{i=0}^{n} \omega_{i,x_i}.$$
- Among the $(2d)^n$ paths, choose one randomly with probability proportional to $e^{\beta\omega(p)}$, where $\beta \geq 0$ is some given parameter.
- The path so chosen is called a directed polymer of length n, at inverse temperature β, in the random environment induced by the random weights.
- When $\beta = 0$, this is just simple symmetric random walk.
- However, when $\beta > 0$, the walk is no longer Markovian. In fact, there is no natural way to connect the walk of length n with the walk of length $n + 1$.

The endpoint distribution

- Let $S_0, \ldots, S_n$ be a directed polymer path of length n.
- The law of S_n (conditional on the environment), which we will denote by f_n, is called the endpoint distribution at time n. This is the central object of interest in this talk.
- Note that f_n is a random probability measure on $\mathbb{Z}^d$.
- Unlike RWRE and ERRW, the endpoint distribution of directed polymers is not expected to converge to any deterministic or random limit as $n \to \infty$.
- Indeed, simulations show that f_n tends to fluctuate wildly, even with small changes in n.

Free energy and phase transition

- Henceforth, assume that the weights are non-degenerate and have finite moment generating function ϕ. Let $\lambda := \log \phi$.
- The free energy at time n is defined as

$$F_n(\beta) = \frac{1}{n} \log \sum e^{\beta \omega(p)},$$

where the sum is over all paths of length n starting at 0.
- The limiting free energy $F(\beta) = \lim_{n\to\infty} F_n(\beta)$ exists and is non-random.
- Comets and Yoshida (2006) proved that in any dimension, there exists $\beta_c \in [0, \infty)$ such that

$$F(\beta) \begin{cases} = \lambda(\beta) & \text{if } 0 \le \beta \le \beta_c, \\ < \lambda(\beta) & \text{if } \beta > \beta_c. \end{cases}$$

- The region $0 \le \beta \le \beta_c$ is called the high temperature phase and the region $\beta > \beta_c$ is called the low temperature phase.

Weak localization

- Let p_n be the mass of the largest atom of the endpoint distribution f_n.
- It follows from results of Carmona and Hu (2002) and Comets, Shiga and Yoshida (2003), that almost surely,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} p_i = 0 \quad \text{if } 0 \le \beta \le \beta_c,$$

  and

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} p_i > 0 \quad \text{if } \beta > \beta_c.$$

- This shows delocalization at high temperature, and a weak form of localization at low temperature (existence of at least one non-vanishing atom).

Our main result

- Recall that the lower density of a set A of nonnegative integers is defined as

$$\liminf_{n\to\infty} \frac{|A \cap \{0, 1, \ldots, n-1\}|}{n}.$$

- We will say that a probability measure on $\mathbb{Z}^d$ is (δ, K)-localized if it puts mass $> 1 - \delta$ on a set of diameter $\leq K$.

Theorem (Bates & C., 2016)

Subsequential strong localization: *For any $\delta > 0$, there exist $K > 0$ and $\theta > 0$ such that the set of all n for which the endpoint distribution f_n is (δ, K)-localized has lower density $\geq \theta$ with probability one.*

Remark: If the theorem can be proved with $\theta = 1$, that would resolve the strong localization conjecture for directed polymers.

Proof

- At its heart, the proof is an application of Jensen's inequality.
- However, Jensen's inequality comes after many steps.
- In particular, we need to build an elaborate abstract machinery, in a new space with a new topology.
- I will now give an outline of these steps.
- Hopefully, the ideas and the abstract tools will be useful for other problems in the future.

Partitioned subprobability measures

- A sequence of probability measures on $\mathbb{Z}^d$ can have some of the mass "diffuse", and the rest clump into "blobs" that "fly away" from each other.
- Suggests that the asymptotic structure can be captured by a subprobability measure f on $\mathbb{N} \times \mathbb{Z}^d$, where the restriction of f to each $\{k\} \times \mathbb{Z}^d$ represents a "blob".
- Let S be the set of all such subprobability measures. We define an equivalence relation on S, and a metric ρ on the set $\mathcal{S}$ of equivalence classes (called partitioned subprobability measures), such that $(\mathcal{S}, \rho)$ is compact.
- Inspired by the method of concentration compactness developed by P.-L. Lions in the 80's, who built on an idea introduced by Parthasarathy, Ranga Rao & Varadhan (1962).
- A similar construction was carried out in $\mathbb{R}^d$ by Mukherjee and Varadhan (2014), but with a different metric. We show that the two metrics are equivalent.

The empirical measure of endpoint distributions: A basic convergence result

- It is not difficult to show that the sequence of endpoint distributions $(f_n)_{n\geq 0}$ is a Markov chain on the set of probability measures on $\mathbb{Z}^d$, and hence on $\mathcal{S}$.
- Let $\mathcal{T}$ denote the transition kernel of this Markov chain.
- $\mathcal{T}$ acts on $\mathcal{P}(\mathcal{S})$, the space of probability measures on $\mathcal{S}$ equipped with the L^1 Wasserstein metric W induced by the metric ρ on $\mathcal{S}$.
- Since $\mathcal{S}$ is compact, there is a nonempty compact set $\mathcal{K} \subseteq \mathcal{P}(\mathcal{S})$ of invariant measures of $\mathcal{T}$.
- Let $\mu_n \in \mathcal{P}(\mathcal{S})$ be the empirical measure of the first n endpoint distributions, that is, $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f_i}$.

Theorem (Bates & C., 2016)

Let $W(\mu_n, \mathcal{K}) := \inf\{W(\mu_n, \mu) : \mu \in \mathcal{K}\}$. *Then* $W(\mu_n, \mathcal{K}) \to 0$ *almost surely as* $n \to \infty$.

Proof idea

- The proof of this theorem uses martingales and the Burkholder–Davis–Gundy inequality.
- It is a special case of the more general heuristic that the empirical measure of a Markov chain run up to time n must be close to an invariant measure of the chain.
- Actually, we prove something stronger: There is a compact subset $\mathcal{M} \subseteq \mathcal{K}$ with some special properties, such that $W(\mu_n, \mathcal{M}) \to 0$ almost surely.
- $\mathcal{M}$ is the set of minimizers of a certain continuous linear functional $\mathcal{R}$ on $\mathcal{P}(\mathcal{S})$.
- $\mathcal{R}$ is complicated to write down. Let me just say that the free energy F_n and the empirical measure μ_n of the endpoint distributions up to time n are related by

$$F_n = \mathcal{R}(\mu_n).$$

(There is a natural way to write F_n as a linear function of μ_n, and $\mathcal{R}$ can be read off from that.)

How to use the convergence theorem

- Let $\mathcal{V}$ be an open subset of $\mathcal{S}$. Suppose that we want to show that $f_n \in \mathcal{V}$ for infinitely many n almost surely.
- First, we prove that $\inf_{\mu \in \mathcal{M}} \mu(\mathcal{V}) = \theta > 0$.
- Since $W(\mu_n, \mathcal{M}) \to 0$ a.s., the compactness of $\mathcal{M}$ implies that any convergent subsequence of μ_n must converge to some $\mu \in \mathcal{M}$.
- Along such a subsequence,

$$\liminf \mu_n(\mathcal{V}) \geq \mu(\mathcal{V}) \geq \theta > 0,$$

  since $\mathcal{V}$ is open.
- Since any subsequence of μ_n contains a convergent subsequence (by compactness of $\mathcal{P}(\mathcal{S})$), this shows that the above inequality holds along the full sequence.
- Finally, since $\mu_n(\mathcal{V}) = \frac{1}{n}\sum_{i=0}^{n-1} 1_{\{f_i \in \mathcal{V}\}}$, this proves that $f_i \in \mathcal{V}$ infinitely often.

Towards the proof of the localization theorem

- Let $f : \mathbb{N} \times \mathbb{Z}^d \to [0,1]$ be a partitioned subprobability measure.
- We will say that f is (δ, K)-localized if there is a set $D \subseteq \mathbb{Z}^d$ of diameter $\leq K$ such that for some $n \in \mathbb{N}$,

$$\sum_{x \in D} f(n,x) > 1 - \delta.$$

- Let $\mathcal{V}_{\delta,K}$ be the set of all (δ, K)-localized partitioned subprobability measures.
- Fact: $\mathcal{V}_{\delta,K}$ is an open subset of $\mathcal{S}$.
- Thus, by the previous slide, the following lemma suffices.

Lemma

For any $\delta > 0$, there exists K such that

$$\inf_{\mu \in \mathcal{M}} \mu(\mathcal{V}_{\delta,K}) > 0.$$

Proof of localization, continued

- Using the compactness of $\mathcal{M}$ and the lower semi-continuity of the map $\mu \mapsto \mu(\mathcal{V}_{\delta,K})$, it is possible to reduce this lemma to the simpler claim that for any $\delta > 0$ and any $\mu \in \mathcal{M}$, there exists K such that $\mu(\mathcal{V}_{\delta,K}) > 0$.
- For $f \in \mathcal{S}$ and $n \in \mathbb{N}$, let

$$q_n(f) := \sum_{x \in \mathbb{Z}^d} f(n, x).$$

- Let $\mathcal{U}_\delta$ be the set of all f such that $q_n(f) > 1 - \delta$ for some n.
- Notice that

$$\mathcal{U}_\delta = \bigcup_{K=0}^{\infty} \mathcal{V}_{\delta,K}.$$

- Thus, we only need to show that for any $\delta > 0$ and $\mu \in \mathcal{M}$, $\mu(\mathcal{U}_\delta) > 0$.

Completing the proof

- Recall that $\mathcal{T}$ is the transition kernel of the Markov chain of endpoint distributions.
- For $f \in \mathcal{S}$, let
$$Q(f) := \sum_{n \in \mathbb{N}} \frac{q_n(f)}{1 - q_n(f)}.$$
- For $\mu \in \mathcal{P}(\mathcal{S})$, let $\mu(Q) := \int Q(f) d\mu(f)$.
- If $\mu \in \mathcal{K}$, then $\mathcal{T}\mu(Q) = \mu(Q)$.
- On the other hand, an argument using Jensen's inequality and the non-degeneracy of weights shows that for any $\mu \neq \delta_0$, either $\mu(Q) = \infty$ or $\mathcal{T}\mu(Q) > \mu(Q)$.
- Thus, $\mu(Q) = \infty$ for all $\mu \in \mathcal{K} \setminus \{\delta_0\}$.
- A similar argument by Jensen's inequality shows that $\delta_0 \notin \mathcal{M}$ when $\beta > \beta_c$.
- Lastly observe that if $\mu(\mathcal{U}_\delta) = 0$ then $Q(f) \leq (1 - \delta)^{-1}$.
- Thus, $\mu(\mathcal{U}_\delta) > 0$ for all $\mu \in \mathcal{M}$ and $\delta > 0$.

Other conclusions

- Besides subsequential strong localization and convergence of the empirical measure of endpoint distributions, the abstract machinery developed here gives a number of other results.
- For example, we can prove the asymptotic pure atomicity of the endpoint distribution at low temperature. Previously, this was proved for heavy-tailed environments by Vargas (2007).
- We also derive a variational formula for the limiting free energy, akin to variational formulas in other disordered systems such as spin glasses. The limiting empirical measure of endpoint distributions is the "order parameter".

Summary

- The main result is a subsequential localization theorem for the endpoint of directed polymers.
- To prove this, we defined a new compactification of probability measures on $\mathbb{Z}^d$, intended to capture localized masses escaping to infinity. We need this because we do not know a priori that the endpoint distribution is localized.
- In the set of probability measures on this compactified space, we identified the set of all possible limit points of the empirical measure of the first n endpoint distributions as $n \to \infty$.
- The set of possible limit measures was shown to have certain nice properties (using mainly Jensen's inequality), which allowed us to deduce the claimed localization theorem.
- Many open questions. Foremost: Prove complete (not subsequential) localization.
- Manuscript on arXiv.