Constructing a solution of the 2D KPZ equation
Infosys-ICTS Ramanujan Lectures: Lecture 3

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Random surfaces

- In modern probability theory, we have a vast and deep understanding of randomly moving points.
- But we are only beginning to understand randomly moving objects of dimension greater than zero. These are generally called random surfaces, although a ‘surface’ should have dimension 2, strictly speaking.
A key objective in probability theory is to show that discrete processes have continuous scaling limits.

Roughly speaking, we consider the discrete process on a scaled lattice like $\epsilon\mathbb{Z}^d$, and look at its limiting behavior as $\epsilon \to 0$.

Often, these limits are universal, in the sense that many different discrete processes have the same scaling limit.

The primary example is convergence of various discrete random walks to Brownian motion.

Another example is Schramm–Loewner evolution (SLE). Many types of self-avoiding random walks and other complicated discrete processes have been shown to converge to SLE in the scaling limit, leading to solutions of longstanding problems.
Just like Brownian motion and SLE are universal scaling limits for a wide variety of discrete random processes, there is supposed to be a universal limit for growing discrete random surfaces.

Here ‘surface’ may be $d$-dimensional for any positive integer $d$.

This limit process is described by the $(d + 1)$-dimensional Kardar–Parisi–Zhang (KPZ) equation.

However, our knowledge about convergence to KPZ is much more limited than our understanding of convergence to Brownian motion or SLE.

Rigorous proofs of convergence exists only for a few classes of one-dimensional surface growth process. (Huge literature.)

For $d \geq 2$, even the limit process (i.e. the solution of the KPZ equation) is not well-understood. Questions of convergence can be investigated only after this is settled.
In this talk, I will present the first rigorous construction of a solution of the 2D KPZ equation.

The solution is obtained by a particular renormalization scheme. It is possible — indeed, probable — that other solutions may be obtained by other schemes.
The \((d + 1)\)-dimensional Kardar–Parisi–Zhang (KPZ) equation

- Describes a growing random surface \(h(t, x)\) \((t \geq 0, x \in \mathbb{R}^d)\):

\[
\partial_t h = \nu \Delta h + \frac{\lambda}{2} |\nabla h|^2 + \sqrt{D} \dot{W}
\]

where \(\nu, \lambda, D\) are positive constants and \(\dot{W}\) is standard space-time white noise.

- Formally, \(\dot{W}\) is a centered Gaussian field satisfying

\[
\mathbb{E}(\dot{W}(t, x)\dot{W}(t', x')) = \delta(t - t')\delta(x - x').
\]

- Rigorously, \(\dot{W}\) is a random distribution with the property that for any smooth \(f\) with compact support, the integral

\[
\int f(t, x)\dot{W}(t, x)dtdx
\]

is a Gaussian random variable with mean zero and variance

\[
\int f(t, x)^2 dtdx.
\]
The KPZ equation was introduced by Kardar, Parisi and Zhang in 1986.

As stated, it is not even clear that the equation is mathematically meaningful.

One key difficulty: $\nabla h$ is expected to be a random distribution, and therefore $|\nabla h|^2$ is undefined.
The case \( d = 1 \)

- Most of the literature is about \( d = 1 \).
- A number of discrete models have been shown to converge to the \((1 + 1)\)-dimensional KPZ equation in the scaling limit, justifying its claim to universality.
- The Cole–Hopf solution of the 1D KPZ equation is obtained as \( h = (2\nu/\lambda) \log u \), where \( u \) solves the stochastic heat equation \((\text{SHE})\) with multiplicative noise:
  \[
  \partial_t u = \nu \partial_x^2 u + (\lambda \sqrt{D}/2\nu) u \dot{W}.
  \]
- This is a solution only ‘in principle’, because it is obtained using Itô’s formula, and Itô’s formula is not applicable here.
- A direct approach to solving the KPZ equation in \( d = 1 \) was provided by Hairer’s theory of regularity structures.
- Other direct approaches: paracontrolled distributions by Gubinelli, Imkeller and Perkowski, energy solutions by Gonçalves and Jara, renormalization approach by Kupiainen.
The solutions to the multiplicative SHE are now well-understood in all dimensions.

The case $d \geq 3$ was analyzed by Mukherjee, Shamov and Zeitouni (2016), and the case $d = 2$ by Caravenna, Sun and Zygouras (2017), and also by Feng (2016), following early contributions of Bertini and Cancrini (1998).

If $u$ is a solution to the $(d + 1)$-dimensional multiplicative SHE, then $h = (2\nu/\lambda) \log u$ is a formal solution of the KPZ equation in any dimension $d$.

However, when $d \geq 2$, $u$ is a distribution rather than a function. So it is not clear how to define $\log u$. 
Special difficulty in $d = 2$

- The $d = 1$ case has a special advantage: In the language of PDEs, it is **subcritical**, and in the language of renormalization, it is **ultraviolet superrenormalizable**.
- All methods for solving the KPZ equation in $d = 1$ depend crucially on the above feature.
- What this means is that there is a way to rescale space and time such that the coefficient of $|\nabla h|^2$ can be made as small as we like while keeping the other two coefficients fixed.
- This is also possible in $d \geq 3$, with a different rescaling. The system is called **infrared superrenormalizable**. Recently analyzed by Magnen and Unterberger (2018).
- The case $d = 2$ is **critical**, in PDE language. In the language of renormalization, it is **not superrenormalizable**.
- To summarize, neither the Cole–Hopf solution nor the direct approaches such as regularity structures are expected to work in $d = 2$. In this sense, $d = 2$ is the hardest dimension.
Approaches to solving the KPZ equation usually begin in the following way.

Let $\dot{W}_\varepsilon$ be the white noise $\dot{W}$ mollified by convolving with a spatial mollifier $\rho^\varepsilon$, where $\rho^\varepsilon(x) = \varepsilon^{-d} \rho(x/\varepsilon)$, and $\rho$ is a compactly supported smooth function.

Formally,

$$\dot{W}^\varepsilon(t, x) = \int \dot{W}(t, y) \rho^\varepsilon(x - y) dy.$$

Consider the mollified equation

$$\partial_t h_\varepsilon = \nu \Delta h_\varepsilon + \frac{\lambda}{2} |\nabla h_\varepsilon|^2 + \sqrt{D} \dot{W}_\varepsilon.$$

Usually, this equation is solvable in a traditional sense.

The question is, what happens as $\varepsilon \to 0$?
The renormalization viewpoint: The parameters $\nu$, $\lambda$ and $D$ need not be fixed parameters; they are free to depend on $\varepsilon$. The main quest is to vary them in such a way so as to get a meaningful/interesting limit object as $\varepsilon \to 0$.

Additionally, for the KPZ equation, one may need to subtract off a constant $C_\varepsilon$ depending on $\varepsilon$.

In the context of making numerical predictions, "meaningful/interesting" means "yields numbers that match experiment".
To summarize, we need to consider the equation

$$\partial_t h_\varepsilon = \nu_\varepsilon \Delta h_\varepsilon + \frac{\lambda_\varepsilon}{2} |\nabla h_\varepsilon|^2 + \sqrt{D_\varepsilon} \dot{W}_\varepsilon - C_\varepsilon,$$

where $\nu_\varepsilon$, $\lambda_\varepsilon$, $D_\varepsilon$ and $C_\varepsilon$ are constants depending on $\varepsilon$ defined in such a way that the solution $h_\varepsilon$ converges to some interesting limit as $\varepsilon \to 0$.

Note that there may be multiple ways to vary $\nu_\varepsilon$, $\lambda_\varepsilon$, $D_\varepsilon$ and $C_\varepsilon$ with $\varepsilon$ that give interesting limits.

When $d = 1$, it turns out that an interesting limit can be obtained if $\nu$, $\lambda$ and $D$ are kept fixed and $C_\varepsilon$ is made to blow up appropriately as $\varepsilon \to 0$ (so that $h_\varepsilon$ always has mean zero).

In fact this is the Cole–Hopf solution in $d = 1$.

In $d \geq 3$, Magnen and Unterberger (2018) scaled $\lambda_\varepsilon$ like $\varepsilon^{(d-2)/2}$, blew up $C_\varepsilon$ appropriately, and kept $\nu$ and $D$ fixed to get a Gaussian scaling limit.
Our result

Theorem (C. and Dunlap, 2018)

Consider the mollified KPZ equation on a 2D torus. Suppose that we keep \( \nu \) and \( D \) fixed, blow up \( C_\varepsilon \) appropriately, and set
\[
\lambda_\varepsilon = \lambda |\log \varepsilon|^{-1/2}
\]
for some \( \lambda > 0 \). That is, consider
\[
\partial_t h_\varepsilon = \nu \Delta h_\varepsilon + \frac{\lambda}{2 \sqrt{\log(1/\varepsilon)}} |\nabla h_\varepsilon|^2 + \sqrt{D} \dot{W}_\varepsilon - C_\varepsilon.
\]

If \( \lambda \) is small enough, the solution \( h_\varepsilon \) converges in law to a limiting random distribution along some sequence of \( \varepsilon \to 0 \). Moreover, this limit is not the same as the one obtained by simply putting \( \lambda = 0 \).
The Family–Vicsek scaling

- The above theorem is the first rigorous result about a scaling limit for 2D KPZ.
- However, our renormalization of parameters is probably not the only possible renormalization that gives an interesting limit.
- Numerical simulations suggest that it may also be possible to obtain a function-valued scaling limit by taking $\nu \sim \varepsilon^{2-z}$, $\lambda \sim \varepsilon^{2-z-\alpha}$, and $D \sim \varepsilon^{2+2\alpha-z}$ for certain exponents $\alpha$ and $z$. This is known as the Family–Vicsek scaling.
- Scaling arguments suggest that $\alpha + z = 2$. If we assume this, then we obtain the scaling $\nu \sim \varepsilon^\alpha$, $\lambda \sim 1$, and $D \sim \varepsilon^{3\alpha}$.
- This amounts to considering the KPZ equation with fixed values of the parameters, and considering the solution multiplied by $\varepsilon^\alpha$ on a short time scale $t \sim \varepsilon^\alpha$.
- There is no consensus about the value of $\alpha$. Mathematically, the problem is wide open.
Our theorem is essentially a tightness result.

We prove tightness of \( \{h_\varepsilon\}_{\varepsilon>0} \) on an appropriate space of distributions.

This is a kind of Besov space, used previously by Hairer and others in the study of KPZ and other systems.

A convenient criterion for tightness on these spaces was recently provided by Furlan and Mourrat (2017), which we use.
Proof sketch

- For simplicity, let me just sketch the proof of tightness for the family of random variables

\[ X_\varepsilon := \int h_\varepsilon \phi, \]

where \( \phi \) is some compactly supported smooth function on the torus.

- Recall that we have adjusted the renormalization constant \( C_\varepsilon \) suitably, so that \( \mathbb{E}(h_\varepsilon(t,x)) = 0 \) for all \( t, x \). Thus, \( \mathbb{E}(X_\varepsilon) = 0 \).

- So it suffices to prove that \( \sup_{0 < \varepsilon < 1} \mathbb{E}(X_\varepsilon^2) < \infty \).
Proving tightness

- The first step is to write $h_\varepsilon$ as $\log u_\varepsilon$, where $u_\varepsilon$ is the solution of the mollified multiplicative SHE.

- Thus,

$$X_\varepsilon = \int \phi \log u_\varepsilon.$$  

- $u_\varepsilon$ has a representation in terms of a Feynman–Kac formula.

- Due to this explicit representation, we can use Malliavin calculus to calculate

$$\frac{\partial}{\partial \lambda} \mathbb{E}(X_\varepsilon^2).$$

- If we can show that the above derivative is uniformly bounded when $0 < \varepsilon < 1$ and $0 \leq \lambda \leq \lambda_0$ for some $\lambda_0 > 0$, and also show that $\sup_{0<\varepsilon<1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $\lambda = 0$, then we will reach our objective of showing that $\sup_{0<\varepsilon<1} \mathbb{E}(X_\varepsilon^2) < \infty$ when $0 \leq \lambda \leq \lambda_0$. 

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Recursion

- Unfortunately, the computation of $\frac{\partial \mathbb{E}(X_\epsilon^2)}{\partial \lambda}$ yields an expectation of the intersection time of two random paths chosen according to a continuum polymer measure, which is Wiener measure with a random tilt.

- We do not know how to calculate expectations of such intersection times.

- But when $\lambda = 0$, the polymer measure reduces to just 2D Wiener measure. We know how to calculate expectations of Brownian intersection times.

- Thus, we can again take derivative with respect to $\lambda$, and hope to get a bound for $\frac{\partial^2 \mathbb{E}(X_\epsilon^2)}{\partial \lambda^2}$.

- But this yields an expectation of a polynomial in the pairwise intersection times of four random paths from a polymer measure.

- One can keep doing this, but it just gets more and more complicated. In $k$ steps, complexity grows like $e^{Ck^2}$.

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To solve the recursion problem, the first step is to reparametrize $\lambda$ as $\sqrt{\beta}$, and take derivatives with respect to $\beta$. Fixing $\varepsilon$, let $g(\beta) := \mathbb{E}(X^2_\varepsilon)$.

We want to control $g(\beta)$ by controlling $g'(\beta)$, and control $g'(\beta)$ by controlling $g''(\beta)$, and so on.

The problem is that these derivatives become hopelessly complicated.

So we find a way around this problem:

- Set $g_0 = g$. Instead of bounding $g'$, find a nicer function $g_1$ such that $|g'(\beta)| \leq g_1(\beta)$.
- To bound $g_1$, we seek to bound $g'_1$. So we find a nicer function $g_2$ such that $|g'_1(\beta)| \leq g_2(\beta)$.
- We keep going like this, and end up with a sequence of functions $g_1, g_2, \ldots$ such that $|g'_k(\beta)| \leq g_{k+1}(\beta)$ for each $k$.
- The nice feature of this sequence is that the complexity grows like $e^{Ck}$. 
Completing the proof

- For each \( k \), \( g_k(0) \) can be obtained by a Brownian computation.
- Finally,

\[
|g(\beta)| \leq \sum_{k=0}^{\infty} \frac{\beta^k}{k!} |g_k(0)|.
\]

- Since the complexity of \( g_k \) increases only exponentially, and the \( k \)th moment of intersection times grow like \( k! \), we can bound \( |g_k(0)| \leq C^k k! \) for some \( C \).
- Thus, for \( \beta \) sufficiently small, we get a finite bound.
- Moreover, it turns out that the bound does not blow up as \( \varepsilon \to 0 \).
The last sentence in our theorem asserts that the scaling limit we obtain is not the same as the one obtained by putting $\lambda = 0$.

In other words, the nonlinear term $|\nabla h|^2$ has a significant effect on the solution.

To prove this, we consider a kind of Fourier decomposition of the solution, and show that high Fourier coefficients have a nonzero contribution, whereas their contribution is zero if $\lambda = 0$. 
Very recent developments

- We posted our paper in September 2018.
- In December 2018, there was a flurry of new advances:
  - Caravenna, Sun and Zygouras posted a manuscript where they show that the scaling limit exists for all $\lambda \in (0, \lambda_c)$, where $\lambda_c$ is possibly the correct critical value. (Recall that $\lambda_\varepsilon = \lambda |\log \varepsilon|^{-1/2}$ is the coefficient of $\frac{1}{2} |\nabla h_\varepsilon|^2$.)
  - Moreover, they also show that the scaling limit is Gaussian, and in fact has the same law as the solution of a stochastic heat equation with additive noise, which is not the same as the one obtained by putting $\lambda = 0$. In physics, this is known as the Edwards–Wilkinson limit.
  - Gu posted a paper with a similar result, but not covering the entire subcritical regime.
  - Dunlap, Gu, Ryzhik and Zeitouni posted a paper containing a new and much shorter proof of the Magnen–Unterberberger existence theorem for the solution of the KPZ equation in $d \geq 3$. 
What happens at criticality?

Renormalization group arguments suggest that instead of taking $\lambda = \lambda_c$, if we use $\lambda = \lambda_c + \text{correction term}$, where the correction term is appropriately vanishing as $\varepsilon \to 0$, it may be possible to construct a non-Gaussian scaling limit.

Getting any non-Gaussian scaling limit for 2D KPZ, either in the above sense, or in the Family–Vicsek scaling, or some other scaling — remains an open problem.

Show that some natural discrete process has a KPZ scaling limit in $d \geq 2$. 