1 Continuation of Stein Bound

In the previous lecture we proved bounds on \( f \) and its derivatives, \( f \) satisfying

\[
f'(x) - xf(x) = g(x) - Ng,
\]

where \( Ng = \mathbb{E} g(Z) \) and \( Z \sim N(0, 1) \).

We are in the process of bounding \( |f''|_\infty \) under the assumption that \( g \) is Lipschitz. Continuing from the previous lecture, we have

\[
f''(x) = g'(x) + (x - \sqrt{2\pi} (1 + x^2) e^{x^2/2} (1 - \Phi(x))) \int_{-\infty}^{x} g'(z) \Phi(z) dz
\]

\[
+ (-x - \sqrt{2\pi} (1 + x^2) e^{x^2/2} \Phi(x)) \int_{x}^{\infty} g'(z) (1 - \Phi(z)) dz.
\]

This gives

\[
|f''(x)|_\infty \leq |g'|_\infty \left[ 1 + |x - \sqrt{2\pi} (1 + x^2) e^{x^2/2} (1 - \Phi(x))| \int_{-\infty}^{x} \Phi(z) dz \right.
\]

\[
+ \left. |x + \sqrt{2\pi} (1 + x^2) e^{x^2/2} \Phi(x)| \int_{x}^{\infty} (1 - \Phi(z)) dz \right].
\]

(1)

Recall the Mill’s ratio inequality on \( \Phi(x) \) for \( x > 0 \):

\[
\frac{xe^{x^2/2}}{\sqrt{2\pi}(1 + x^2)} \leq 1 - \Phi(x) \leq \frac{e^{x^2/2}}{x\sqrt{2\pi}}.
\]

(2)

**Exercise 1** Prove the left inequality in (2).

There is a similar bound for \( x \leq 0 \). To proceed, we wish to remove the absolute values in equation (1), by determining the sign of the expressions within the absolute value. From the Mill’s ratio (2) we have

\[
x + \sqrt{2\pi} (1 + x^2) e^{x^2/2} \Phi(x) > 0
\]

(3a)
and
\[-x + \sqrt{2\pi}(1 + x^2)e^{x^2/2}(1 - \Phi(x)) > 0. \quad (3b)\]
You can check (3b) by noting that for \(x < 0\) the inequality is obvious, and for \(x > 0\) use the lower Mill’s ratio inequality; (3a) follows similarly. Hence both expressions within the absolute values in equation (1) are negative.

To finish the simplification, observe that integration by parts gives
\[
\int_{-\infty}^{x} \Phi(z)dz = x \Phi(x) + \frac{e^{-x^2/2}}{\sqrt{2\pi}}
\]
and
\[
\int_{x}^{\infty} (1 - \Phi(z))dz = -x(1 - \Phi(x)) + \frac{e^{-x^2/2}}{\sqrt{2\pi}}.
\]
Combining, we get
\[
|f''(x)| \leq |g'|_{\infty} \left[ 1 + \left( -x + \sqrt{2\pi}(1 + x^2)e^{x^2/2}(1 - \Phi(x)) \right) \left( x \Phi(x) + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right) \\
+ \left( x + \sqrt{2\pi}(1 + x^2)e^{x^2/2}\Phi(x) \right) \left( -x(1 - \Phi(x)) + \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right) \right] = 2|g'|_{\infty}. \quad (4)
\]
This proves the desired bound.

The factor 2 in the bound above turns out to be sharp. In contrast to the calculation above, it is easy to attain a factor of 4, as follows. First we take the derivative of the equation
\[f'(x) - xf(x) = g(x) - Ng\]
to get
\[f''(x) - xf'(x) - f(x) = g'(x),\]
or
\[f''(x) - xf'(x) = g'(x) + f(x) := h(x).\]
Since \(Nh = Eh(Z) = Ef''(Z) - EZf'(Z) = 0\) (from Stein’s Lemma in lecture 2), we see that \(f'\) is a solution of the Stein equation with \(h\). The triangle inequality and one of the earlier Stein bounds give
\[|g' + f| \leq |g'| + |f| \leq 2|g'|_{\infty},\]
hence
\[|f''|_{\infty} \leq 2|g' + f|_{\infty} \leq 4|g'|_{\infty}.
\]
This completes the discussion of the five Stein bounds.
2 Dependency Graphs

Let \( \{X_i, i \in V\} \) be a collection of random variables, and \( G = (V, E) \) be a graph with vertex set \( V \).

**Definition 2** \( G \) is called a dependency graph for \( \{X_i, i \in V\} \) if the following holds: for any two subsets of vertices \( S, T \subseteq V \) such that there is no edge from any vertex in \( S \) to any vertex in \( T \), the collections \( \{X_i, i \in S\} \) and \( \{X_i, i \in T\} \) are independent.

The idea behind the usefulness of dependency graphs is that if the max degree is not too large, we get a CLT. Note that there is not a unique dependency graph (for example, the complete graph works for any set of r.v.).

**Example 3** Suppose \( Y_1, Y_2, \ldots, Y_{n+1} \) are independent random variables, and let \( X_i = Y_i Y_{i+1} \). We will want to study the behavior of

\[
\sum_i X_i = \sum_i Y_i Y_{i+1}.
\]

A dependency graph for \( \{X_i, i \in V\} \) with \( V = \{1, \ldots, n\} \) is given by the graph with edge set \( \{(i, i+1); 1 \leq i \leq n-1\} \).

Given a graph \( G \), let \( D = 1 + \) maximum degree of \( G \). We have the following lemma.

**Lemma 4** Let \( S = \sum_{i \in V} X_i \). Then \( \text{Var}(S) \leq D \sum_{i \in V} \text{Var}(X_i) \).

**Proof:** Assume without loss of generality that \( \mathbb{E}(X_i) = 0 \) for all \( i \in V \). We write \( j \sim i \) if \( j \) is a neighbor of \( i \) or \( j = i \). Then

\[
\text{Var}(S) = \sum_{i,j} \mathbb{E}(X_i X_j) \overset{(a)}{=} \sum_{i,j \sim i} \mathbb{E}(X_i X_j) \overset{(b)}{\leq} \sum_{i,j \sim i} \frac{\mathbb{E} X_i^2 + \mathbb{E} X_j^2}{2} \leq D \sum_{i \in V} \text{Var}(X_i),
\]

where (a) follows by the zero-mean assumption and independence, and (b) from the AM-GM inequality \( ab \leq (a^2 + b^2)/2 \). \( \square \)

In the next lecture we will use the lemma to prove the following theorem. Let \( \sigma^2 = \text{Var}(\sum X_i) \) and \( W = \sum X_i / \sigma \), where it is assumed that \( \mathbb{E}(X_i) = 0 \).

**Theorem 5** It holds that

\[
\text{Wass}(W, Z) \leq \frac{4}{\sqrt{\pi} \sigma^2} \sqrt{D^3 \sum \mathbb{E} |X_i|^4 + \frac{D^2}{\sigma^3} \sum \mathbb{E} |X_i|^3},
\]

where \( Z \sim N(0, 1) \).
Remark 6 The bound in the theorem is often tight. We can get a bound on the Kolmogorov metric from the bound

$$\text{Kolm}(W, Z) \leq \frac{2}{(2\pi)^{1/4}} \sqrt{\text{Wass}(W, Z)},$$

but this is not a good bound.