In this lecture we are going to study the solution of the differential equation
\[ f'(x) - xf(x) = g(x) - Eg(Z), \quad Z \sim N(0,1). \quad (1) \]

**Lemma 1** Given function \( g : \mathbb{R} \to \mathbb{R} \) such that \( E|g(Z)| < \infty \) where \( Z \sim N(0,1) \),

\[ f(x) = e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} (g(y) - \text{E}g(Z)) \, dy \quad (2) \]

is an absolutely continuous solution of (1).

Moreover, any a.c. solution \( \tilde{f} \) of (1) is of the form

\[ \tilde{f}(x) = f(x) + ce^{x^2/2}, \quad c \in \mathbb{R}. \]

Finally, \( f \) is the only solution that satisfies \( \lim_{|x| \to \infty} f(x) e^{-x^2/2} = 0 \).

**Proof:** By the method of integrating factors, we have that if \( f \) is a solution to (1), then

\[ \frac{d}{dx} \left( e^{-x^2/2} f(x) \right) = e^{-x^2/2} \left( f'(x) - xf(x) \right) = e^{-x^2/2} (g(x) - \text{E}g(Z)). \]

So, (2) is a reasonable candidate as a solution of (1). And it is easy to verify directly that (2) indeed satisfies (1).

If \( \tilde{f} \) is any other solution of (1), then

\[ \frac{d}{dx} \left( e^{-x^2/2} (f(x) - \tilde{f}(x)) \right) = 0. \]

Hence, \( \tilde{f}(x) = f(x) + ce^{x^2/2} \) for some \( c \in \mathbb{R} \).

Clearly, from definition

\[ \lim_{x \to -\infty} f(x) e^{-x^2/2} = 0 \quad (\text{by DCT}). \]

Note that since \( Z \sim N(0,1) \), we have

\[ \int_{-\infty}^{\infty} e^{-y^2/2} (g(y) - \text{E}g(Z)) \, dy = 0. \]
So, $f$ can also be written as follows

$$f(x) = -e^{x^2/2} \int_x^{\infty} e^{-y^2/2} (g(y) - \mathbb{E} g(Z)) dy. \quad (3)$$

Therefore, by DCT, $\lim_{x \to +\infty} f(x) e^{-x^2/2} = 0$.

□

**Remark 2** If, instead of standard gaussian, $Z$ follows any other distribution then all of the statements of the above lemma still hold except $\lim_{x \to +\infty} f(x) e^{-x^2/2} = 0$.

### 0.1 Another form of the solution

**Lemma 3** Assume $g$ is Lipschitz. Then

$$f(x) = -\int_0^1 \frac{1}{2\sqrt{1-t}} \mathbb{E} \left[ Zg(\sqrt{t}x + \sqrt{1-t}Z) \right] dt, \quad Z \sim N(0, 1) \quad (4)$$

is a solution of (1). In fact, it must be the same as (2), because $\lim_{|x| \to \infty} f(x) e^{-x^2/2} = 0$.

**Proof:** Let $g$ is $C$-Lipschitz. Then $^1 |g'|_\infty \leq C$.

On differentiating $f$ and carrying the derivative inside the integral and expectation which can be justified using DCT, we have

$$f'(x) = -\int_0^1 \frac{1}{2\sqrt{1-t}} \mathbb{E} \left[ Zg'(\sqrt{t}x + \sqrt{1-t}Z) \right] dt. \quad (5)$$

On the other hand, the Stein identity gives us

$$\mathbb{E} \left[ Zg(\sqrt{t}x + \sqrt{1-t}Z) \right] = \sqrt{1-t} \mathbb{E} \left[ g'(\sqrt{t}x + \sqrt{1-t}Z) \right].$$

Thus,

$$f'(x) - xf(x) = \int_0^1 \mathbb{E} \left[ \left( -\frac{Z}{2\sqrt{1-t}} + \frac{x}{2\sqrt{t}} \right) g'(\sqrt{t}x + \sqrt{1-t}Z) \right] dt$$

$$= \int_0^1 \mathbb{E} \left[ \frac{d}{dt} g'(\sqrt{t}x + \sqrt{1-t}Z) \right] dt$$

$$= \mathbb{E} \left[ \int_0^1 \frac{d}{dt} g'(\sqrt{t}x + \sqrt{1-t}Z) dt \right] = g(x) - \mathbb{E} g(Z).$$

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^1 Any Lipschitz function $g$ is absolutely continuous. Hence, it is (Lebesgue) almost surely differentiable. Define $g'$ to be derivative of $g$ at the points where it exists and 0 elsewhere.
Recall the notation $N g := E g(Z)$. Now we will prove that if $g : \mathbb{R} \to \mathbb{R}$ is bounded,

$I$. $|f|_\infty \leq \sqrt{\frac{\pi}{2}} |g - N g|_\infty$ and $II$. $|f'|_\infty \leq 2 |g - N g|_\infty$

and if $g$ is Lipschitz, but not necessarily bounded, then

$III$. $|f|_\infty \leq |g'|_\infty$, $IV$. $|f'|_\infty \leq \sqrt{\frac{2}{\pi}} |g'|_\infty$, and $V$. $|f''|_\infty \leq 2 |g'|_\infty$.

This will prove the Lemma 1 of Lecture 3. The bounds (I), (II) and (V) were obtained by Stein.

**Proof of bound (III)**: Applying Stein’s identity on (4), we have

$$f(x) = - \int_0^1 \frac{1}{2\sqrt{t}} E \left[ g'(\sqrt{t}x + \sqrt{1-t}Z) \right] dt.$$

Hence,

$$|f|_\infty \leq |g'|_\infty \int_0^1 \frac{1}{2\sqrt{t}} = |g'|_\infty.$$

**Proof of bound (IV)**: From (5), it follows that

$$|f|_\infty \leq (E|Z|) |g'|_\infty \int_0^1 \frac{1}{2\sqrt{1-t}} = \sqrt{\frac{2}{\pi}} |g'|_\infty.$$

**Exercise 4** Get the bound (V) from the representation (4).

**Proof of bound (I)**: Take $f$ as in (2). Suppose $x > 0$. Using the representation in (3), we have

$$|f(x)| \leq |g - N g|_\infty \left( e^{x^2/2} \int_x^\infty e^{-y^2/2} dy \right).$$

Now, $\frac{d}{dx} e^{x^2/2} \int_x^\infty e^{-y^2/2} dy = -1 + xe^{x^2/2} \int_x^\infty e^{-y^2/2} dy \leq 0 \ \forall x > 0$. The last step follows from Mill’s ratio inequality which says that $\int_x^\infty e^{-y^2/2} dy \leq \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (for a quick proof, note that LHS $\leq \int_x^\infty \frac{y}{x} e^{-y^2/2} dy = \text{RHS}$).

So, $e^{x^2/2} \int_x^\infty e^{-y^2/2} dy$ is maximized at $x = 0$ on $[0, \infty)$ where its value is $\sqrt{\frac{\pi}{2}}$. Hence,

$$|f(x)| \leq \sqrt{\frac{\pi}{2}} |g - N g|_\infty \ \forall x > 0.$$
For $x < 0$, use the form (2) and proceed in the similar manner. □

**Proof of bound (II):** Again, we will only consider $x > 0$ case. The other case will be similar.

Note that

$$f'(x) = g(x) - Ng + xf(x) = g(x) - Ng - xe^{x^2/2} \int_{x}^{\infty} e^{-y^2/2}(g(y) - Ng)dy.$$ 

Therefore,

$$|f'(x)| \leq |g - Ng|_\infty \left(1 + xe^{x^2/2} \int_{x}^{\infty} e^{-y^2/2}dy\right)$$

$$\leq 2|g - Ng|_\infty \quad \text{(By Mill’s ratio inequality)}.\]

**Proof of bound (V):** On differentiating (1) and rearranging

$$f''(x) = g'(x) + f(x) + xf'(x)$$

$$= g'(x) + f(x) + x(g(x) - Ng + xf(x))$$

$$= g'(x) + x(g(x) - Ng) + (1 + x^2)f(x). \quad (6)$$

We can write $g(x) -Ng$ in terms of $g'$ as follows,

$$g(x) - Ng = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}(g(x) - g(y))dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{x} \int_{y}^{\infty} g'(z)e^{-y^2/2}dzdy - \int_{x}^{\infty} \int_{x}^{y} g'(z)e^{-y^2/2}dzdy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{x} g'(z) \int_{y}^{\infty} e^{-y^2/2}dydz - \int_{x}^{\infty} g'(z) \int_{x}^{\infty} e^{-y^2/2}dydz \right]$$

$$= \int_{-\infty}^{x} g'(z)\Phi(z)dz - \int_{x}^{\infty} g'(z)\overline{\Phi}(z)dz$$

where $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}dy$ is the distribution function for standard normal and $\overline{\Phi}(z) = 1 - \Phi(z)$.  

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Similarly,

\[ f(x) = e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} (g(y) - \mathbb{E}g(Z)) dy \]

\[ = e^{x^2/2} \int_{-\infty}^{x} e^{-y^2/2} \left( \int_{-\infty}^{y} g'(z) \Phi(z) dz - \int_{y}^{\infty} g'(z) \Phi(z) dz \right) dy \]

\[ = e^{x^2/2} \left( \int_{-\infty}^{x} g'(z) \Phi(z) \int_{z}^{x} e^{-y^2/2} dy dz - \int_{-\infty}^{\infty} g'(z) \Phi(z) \int_{-\infty}^{z} e^{-y^2/2} dy dz \right) \]

\[ = \sqrt{2\pi} e^{x^2/2} \left( \int_{-\infty}^{x} g'(z) \Phi(z) (\Phi(z) - \Phi(x)) dz \right. \]

\[ - \int_{-\infty}^{x} g'(z) \Phi(z) \Phi(z) dz - \int_{x}^{\infty} g'(z) \Phi(z) \Phi(x) dz \]

\[ = -\sqrt{2\pi} e^{x^2/2} \left[ \Phi(x) \int_{-\infty}^{x} g'(z) \Phi(z) dz + \Phi(x) \int_{x}^{\infty} g'(z) \Phi(z) dz \right] \]

Substituting the above expressions for \( g - Ng \) and \( f \) in (6), we get

\[ f''(x) = g'(x) + \left( x - \sqrt{2\pi} (1 + x^2) e^{x^2/2} \Phi(x) \right) \int_{-\infty}^{x} g'(z) \Phi(z) dz \]

\[ + \left( -x - \sqrt{2\pi} (1 + x^2) e^{x^2/2} \Phi(x) \right) \int_{x}^{\infty} g'(z) \Phi(z) dz. \]

\[ To \ be \ continued \ in \ the \ next \ lecture. \]