When \( h = 0 \), a phase transition occurs at \( \beta = 1 \). We saw that in the high temperature phase, \( R_{12} = O(N^{-1/2}) \). Parisi conjectures that at \( \beta = 1 \) and \( h = 0 \), \( R_{12} \) is of order \( N^{-1/3} \). Guerra proved that at \( \beta = 1 \) and \( h = 0 \), \( \mathbb{E}\langle R_{12}^2 \rangle \leq \frac{C}{\sqrt{N}} \) for some constant \( C \), ie, \( R_{12} = O(N^{-1/4}) \) at most. Talagrand has another proof in his book, but it is complicated. Nothing better is known.

Using Stein’s method, we will show that \( \mathbb{E}\langle |R_{12}|^3 \rangle \geq \frac{C}{N} \) for some \( C > 0 \).

**Proof:** Let \( \psi(x) \) be the probability density
\[
\frac{\cosh(x)e^{-x^2/2}}{\sqrt{2\pi e}}
\] (i.e. the symmetric mixture of \( N(1,1) \) and \( N(-1,1) \)). Hopefully, this is the distribution of the local field as \( N \to \infty \).

For any bounded, measurable \( f \), let \( Mf = \int_{-\infty}^{\infty} f(x)\psi(x) \, dx \). Define an operator \( U \) as
\[
Uf(x) = \frac{e^{x^2/2}}{\cosh x} \int_{-\infty}^{\infty} \cosh(t)e^{-t^2/2}(f(t) - Mf) \, dt
\]
and let \( Tf(x) = f'(x) - (x - \tanh(x))f(x) \). Verify that \( TUf = f - Mf \), so \( U \) is the inversion of the Stein operator.

**Lemma 1** \( ||Uf||_\infty \leq C||f||_\infty \) and \( ||(Uf)'||_\infty \leq C||f||_\infty \) for some universal constant \( C \).

**Lemma 2 (Expansion lemma)** Fix a bounded, measurable \( f : \mathbb{R} \to \mathbb{R} \), let \( b_1, \ldots, b_m \) be arbitrary functions of \( \sigma \). Assume that \( b_1 \) does not depend on \( \sigma_1 \). Then
\[
\mathbb{E}(\langle f(l_1)b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle) = (Mf)\mathbb{E}(\langle b_1 \rangle \cdots \langle b_m \rangle)
\]
\[
- \sum_{r=2}^{m} \frac{1}{N} \sum_{j=2}^{N} \mathbb{E}(\langle \sigma_j Uf(l_1)b_1 \rangle \langle b_2 \rangle \cdots \langle b_{r-1} \sigma_j \rangle \langle b_{r+1} \rangle \cdots \langle b_m \rangle)
\]
\[
+ \frac{m}{N} \sum_{j=2}^{N} \mathbb{E}(\langle \sigma_j Uf(l_1)b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle) + \frac{1}{N} \mathbb{E}(\langle (Uf)'(l_1)b_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle)
\]
\[
+ \frac{1}{N} \mathbb{E}(\langle \sigma_j Uf(l_1)b_1 \sigma_1 \rangle \langle b_2 \rangle \cdots \langle b_m \rangle)
\]
Now take \( f(x) = \tanh(x) \). Then \( \langle \sigma_1 \sigma_2 \rangle = \langle f(l_1) \sigma_2 \rangle \) (since \( f(l_1) \) is the conditional expectation of \( \sigma_1 \) given the rest of the spins). Thus \( E\langle \sigma_1 \sigma_2 \rangle^2 = E(\langle f(l_1) \sigma_2 \rangle \langle \sigma_1 \sigma_2 \rangle) \).

Let \( b_1 = \sigma_2, b_2 = \sigma_1 \sigma_2 \). Since \( \psi \) is a symmetric density and \( \tanh \) is odd, \( Mf = 0 \). Let \( h = Uf \). Applying the expansion lemma,

\[
E\langle \sigma_1 \sigma_2 \rangle^2 = 0 - \frac{1}{N} \sum_{j=2}^{N} E(\langle \sigma_1 h(l_1) \sigma_2 \rangle \langle \sigma_2 \sigma_j \rangle)
+ \frac{2}{N} \sum_{j=2}^{N} E(\langle \sigma_j h(l_1) \sigma_2 \rangle \langle \sigma_2 \sigma_j \rangle \langle \sigma_1 \sigma_j \rangle)
+ \frac{1}{N} E(\langle \sigma_l h(l_1) \sigma_2 \rangle \langle \sigma_1 \sigma_2 \rangle)
+ \frac{1}{N} E(\langle h(l_1) \sigma_1 \sigma_2 \rangle \langle \sigma_1 \sigma_2 \rangle)
\]

To simplify the above, let \( Rem \) denote any term that is bounded by \( CN^{-1} \sqrt{E(R_{12})^2} \) for some constant \( C \).

Since \( h' \) is bounded, so

\[
\frac{1}{N} E(\langle h'(l_1) \sigma_2 \rangle \langle \sigma_1 \sigma_2 \rangle) \leq \frac{C}{N} E(\langle \sigma_1 \sigma_2 \rangle) \leq \frac{C}{N} \sqrt{E(\sigma_1 \sigma_2)^2}
\]

\[
\leq \frac{C'}{N} \sqrt{\frac{1}{N^2} \sum_{i,j} \langle \sigma_i \sigma_j \rangle^2} = \frac{C'}{N} \sqrt{E(R_{12})^2}.
\]

So the fourth term is \( Rem \). The last term is also \( Rem \). Any single term in the sum in the third term is also \( Rem \). In the 2nd term, for \( j = 2 \), we get \( \frac{1}{N} E(h(l_1)) \). All other terms are \( Rem \). So

\[
E\langle \sigma_1 \sigma_2 \rangle^2 = -\frac{1}{N} E(h(l_1)) - E(\langle h(l_1) \sigma_2 \sigma_3 \rangle \langle \sigma_2 \sigma_3 \rangle) - 2E(\langle h(l_1) \sigma_2 \sigma_3 \rangle \langle \sigma_1 \sigma_2 \rangle \langle \sigma_1 \sigma_3 \rangle) - Rem.
\]

This is the first expansion. To complete the proof, we apply the expansion lemma to each of the above terms. It will be enough to show:

\[
-\frac{E(h(l_1))}{N} = \frac{1}{N} + Rem,
\]

\[
-\frac{E(h(l_1) \sigma_2 \sigma_3)}{N} = E(\sigma_1 \sigma_2)^2 + T_1,
\]

where \( |T_1| \leq CE(\|R_{12}\|^3) \), and

\[
|E(\langle h(l_1) \sigma_2 \sigma_3 \rangle \langle \sigma_1 \sigma_2 \rangle \langle \sigma_1 \sigma_3 \rangle)| \leq CE(\|R_{12}\|^3) + Rem.
\]
If we can show the above, then
\[ E(\sigma_1 \sigma_2)^2 = \frac{1}{N} + E(\sigma_1 \sigma_2)^2 + T_1 + T_2 \]
where \( T_1, T_2 \) are bounded by \( CE(\|R_{12}\|^3) + \text{Rem} \). So
\[ \frac{1}{N} \leq CE(\|R_{12}\|^3) + \text{Rem} \leq CE(\|R_{12}\|^3) + \frac{C' \sqrt{E(\|R_{12}\|^3)}}{N} \]
Now \( \sqrt{E(\|R_{12}\|^3)} \leq (E(\|R_{12}\|^3))^{1/3} \). Suppose \( E(\|R_{12}\|^3) \leq \frac{1}{2CN} \). Then we get \( \frac{1}{N} \leq 1/2N + (C'/N)(1/2CN)^{1/3} \), a contradiction when \( N \) is large enough. So for \( N \) large enough, \( E(\|R_{12}\|^3) \geq \frac{1}{2CN} \).

To prove the above three statements:

Apply the approximation lemma, and only the first terms will matter. Let \( w = Uh \) (so we invert the Stein operator again). Verify that \( Mh = -1 \). By the expansion lemma,
\[ E(h(l_1)) = -1 + \frac{1}{N} \sum_{j=2}^{N} E(\langle \sigma_j w(l_1) \rangle \langle \sigma_1 \sigma_j \rangle) + \frac{E(w'(l_1)) + E(\sigma_1 \sigma_j)}{N} \]
\[ = -1 + \text{Rem} + O(1/N). \]

Using the expansion lemma on the second term, \( E(\langle h(l_1) \sigma_2 \sigma_3 \rangle \langle \sigma_2 \sigma_3 \rangle) = (Mh)E(\sigma_2 \sigma_3)^2 = -E(\sigma_2 \sigma_3)^2 = -E(\sigma_1 \sigma_2)^2 \) with some remainder terms. The third term can also be bounded through the expansion lemma.

Lastly, a sketch of the proof of the expansion lemma:

We want to find \( E(\langle f(l_1)b_1 \cdots b_m \rangle) \). Let \( h = Uf \) so that \( h'(x) - (x - \tanh(x))h(x) = f(x) - Mf \). Replace \( f(l_1) - Mf \) by \( h'(l_1) - l_1 h(l_1) + \tanh(l_1) h(l_1) \) and apply integration by parts on the terms arising from \( l_1 h(l_1) \).

\( \square \)

**Exercise 3** Get an upper bound for \( E(\|R_{12}\|^3) \).

**Exercise 4** Evaluate
\[ \lim_{N \to \infty} E(N \langle R_{12} R_{23} R_{31} \rangle) \]
or, alternatively,
\[ \lim_{N \to \infty} E(N \langle \sigma_1 \sigma_2 \rangle \langle \sigma_2 \sigma_3 \rangle \langle \sigma_3 \sigma_1 \rangle) \]
You can use Guerra’s result.