Recall that, in the Sherrington Kirkpatrick model, the probability of a configuration \( \sigma = (\sigma_i)_{i=1}^N \in \{-1, +1\}^N \) is

\[
P(\sigma) = Z_N^{-1} \exp \left( \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i \right)
\]

where \( (g_{ij})_{1 \leq i < j \leq N} \) are i.i.d. standard gaussian random variables and

\[
Z_N = \sum_{\sigma \in \{-1, +1\}^N} \exp \left( \frac{\beta}{\sqrt{N}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i \right)
\]

is the normalizing constant. Suppose \( \sigma^1, \sigma^2 \) are i.i.d. configurations from this Gibbs measure given \( g = (g_{ij})_{i<j} \). The overlap between \( \sigma^1, \sigma^2 \) is defined as

\[
R_{12} = \frac{1}{N} \sum_{i=1}^N \sigma^1_i \sigma^2_i.
\]

Suppose \( \langle \cdot \rangle \) denotes conditional expectation w.r.t. the Gibbs measure given \( g \) and \( \nu \) denotes unconditional expectation, i.e. \( \nu(f) = E \langle f \rangle \). Then we have the following result.

**Theorem 1** \( \exists \beta_0 > 0 \) such that for all \( \beta \in [0, \beta_0] \) and for all \( h \)

\[
\frac{\log Z_N}{N} \to \log 2 + E \log \cosh(\beta z \sqrt{q} + h) + \frac{\beta^2(1-q)^2}{4}
\]

where \( q \) satisfies \( q = E \tanh^2(\beta z \sqrt{q} + h) \) and \( z \sim N(0,1) \).

**Idea of the proof:** Choose any arbitrary number \( q \in [0, 1] \). Consider the alternative Gibbs measure \( \propto \exp(\sum_{i=1}^N (\beta z_i \sqrt{q} + h) \sigma_i) \) where \( z_1, z_2, \ldots, z_N \) are i.i.d. N(0, 1) random variables independent of \( g \). Let \( \nu_0 \) be the unconditional law of this Gibbs measure. Note that \( \sigma_i \)'s are independent under this Gibbs measure (both conditionally and unconditionally) and this measure is easier to handle. Also

\[
\frac{\beta}{\sqrt{N}} \sum_{1 \leq i < j \leq N} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^N \sigma_i = \sum_{i=1}^N \left( \frac{\beta}{2} \sigma_i^2 + h \right) \sigma_i
\]
where \( l_i = \frac{1}{\sqrt{N}} \sum_{j=1, j \neq i}^{N} g_{ij} \sigma_j \). The main idea is to show that for \( \beta \) sufficiently small, with a proper choice of \( q \) one can compare \( \nu_0 \) and \( \nu \) “in some sense”. In the last lecture we proved that \( N^{-1}(\log Z_N - E \log Z_N) \rightarrow 0 \) in probability. Today we’ll prove that

\[
E \left( \frac{\log Z_N}{N} \right) \leq \log 2 + E \log \cosh(\beta z \sqrt{q} + h) + \frac{\beta^2(1-q)^2}{4} \text{ for all } q \in [0,1], \beta \geq 0, h \in \mathbb{R}.
\]

**Lemma 2 (Gaussian Interpolation)** Suppose \( X = (X_1, \ldots, X_n) \) and \( Y = (Y_1, \ldots, Y_n) \) are two centered gaussian random vectors independent of each other. Let \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) be a \( C^2 \) function and let

\[
\varphi(t) = EF(\sqrt{t}X + \sqrt{1-t}Y).
\]

Then we have

\[
\varphi'(t) = \frac{1}{2} \sum_{i,j=1}^{n} (E(X_iX_j) - E(Y_iY_j)) \cdot E \left( \frac{\partial^2 F}{\partial x_i \partial y_j} \right)(\sqrt{t}X + \sqrt{1-t}Y).
\]

In particular we have

\[
EF(X) - EF(Y) = \int_0^1 \varphi'(t)dt.
\]

**Proof:** Exercise. \( \square \)

For each \( \sigma \in \{-1, +1\}^N \), let

\[
u_\sigma = \beta \sqrt{q} \sum_{i=1}^{N} z_i \sigma_i \]

Then the normalizing constants in the S-K model and in the alternative model are

\[
Z_N = \sum_{\sigma} \exp(u_\sigma + h \sum \sigma_i) \quad \text{and} \quad Z_N^0 = \sum_{\sigma} \exp(v_\sigma + h \sum \sigma_i)
\]

respectively. So if we define a function \( Z : \mathbb{R}^{(-1,+1)^N} \rightarrow \mathbb{R} \) as

\[
Z(x) = \sum_{\sigma \in \{-1, +1\}^N} w_\sigma \exp(x_\sigma)
\]

where \( x = (x_\sigma)_{\sigma \in \{-1, +1\}^N} \) and \( w_\sigma = \exp(h \sum_{i=1}^{N} \sigma_i) \), we have \( Z_N = Z(u), Z_N^0 = Z(v) \) where \( u = \{u_\sigma\}_{\sigma \in \{-1, +1\}^N} \) and \( v = \{v_\sigma\}_{\sigma \in \{-1, +1\}^N} \). Let

\[
F(x) = \frac{\log Z(x)}{N} \text{ for } x \in \mathbb{R}^{\{-1, +1\}^N}
\]
Let $\varphi(t) = EF(\sqrt{t}u + \sqrt{1 - tv})$. We are interested in
\[ \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t) dt. \]

Clearly we have
\[ \frac{\partial F}{\partial x_\sigma} = \frac{\partial}{\partial x_\sigma} \left( \frac{\log Z(x)}{N} \right) = \frac{1}{NZ(x)} w_\sigma \exp(x_\sigma) \]
and
\[ \frac{\partial^2 F}{\partial x_\tau \partial x_\sigma} = -\frac{1}{N(Z(x))^2} w_\tau w_\sigma \exp(x_\sigma + x_\tau) + \frac{1}{NZ(x)} w_\sigma \exp(x_\sigma) \cdot 1_{\{\sigma = \tau\}}. \]

Let $U(\sigma^1, \sigma^2) = \frac{1}{2} E(u_{\sigma^1} u_{\sigma^2} - v_{\sigma^1} v_{\sigma^2})$. Then
\[
\varphi'(t) = \sum_{\sigma^1, \sigma^2} U(\sigma^1, \sigma^2) E \left( \frac{\partial^2 F}{\partial x_{\sigma_1} \partial x_{\sigma_2}} \left( \sqrt{t}u + \sqrt{1 - tv}\right) \right)
= \frac{1}{N} \sum_{\sigma} U(\sigma, \sigma) w_\sigma \exp(\sqrt{t}u_\sigma + \sqrt{1 - tv_\sigma}) \frac{1}{Z_t} \sum_{\sigma^1, \sigma^2} U(\sigma^1, \sigma^2) w_{\sigma^1} w_{\sigma^2} \exp(\sqrt{t}(u_{\sigma^1} + u_{\sigma^2}) + \sqrt{1 - t}(v_{\sigma^1} + v_{\sigma^2})) \frac{1}{Z_t^2}
\]
where $Z_t = \sum_{\sigma} w_\sigma \exp(\sqrt{t}u_\sigma + \sqrt{1 - tv_\sigma})$. For each $t \in [0, 1]$ we have a gibbs measure $\propto \exp(\sqrt{t}u_\sigma + \sqrt{1 - tv_\sigma} + h \sum \sigma_i)$ where $u_\sigma = \beta \sqrt{N} \sum_{i<j} g_{ij} \sigma_i \sigma_j$ and $v_\sigma = \beta \sqrt{q} \sum_{i=1}^N z_i \sigma_i$. Let $\langle \cdot \rangle_t$ denote the expectation w.r.t. this gibbs measure. Let $\nu_t$ denote the unconditional expectation. Then
\[
\varphi'(t) = \frac{1}{N} (E(U(\sigma, \sigma))_t - E(U(\sigma^1, \sigma^2))_t).
\]

Now
\[
U(\sigma^1, \sigma^2) = \frac{\beta^2}{2N} E \left( \left( \sum_{i<j} g_{ij} \sigma_i^1 \sigma_j^1 \right) \left( \sum_{i<j} g_{ij} \sigma_i^2 \sigma_j^2 \right) \right) - \frac{\beta^2 q}{2} E \left( \sum_{i=1}^N Z_i \sigma_i^1 \sum_{i=1}^N Z_i \sigma_i^2 \right)
= \frac{\beta^2}{2N} \sum_{i<j} \sigma_i^1 \sigma_j^1 \sigma_i^2 \sigma_j^2 - \frac{\beta^2 q}{2} \sum_{i=1}^N \sigma_i^1 \sigma_i^2
= \frac{\beta^2}{4N} \left( \sum_{i=1}^N \sigma_i^1 \sigma_i^2 \right)^2 - \frac{\beta^2 q}{2} \sum_{i=1}^N \sigma_i^1 \sigma_i^2
= \frac{\beta^2 N}{4} \left( R_{12}^2 - \frac{1}{N} \right) - \frac{\beta^2 q N}{2} R_{12}
\implies \frac{1}{N} U(\sigma^1, \sigma^2) = \frac{\beta^2}{4} \left( R_{12}^2 - \frac{1}{N} \right) - \frac{\beta^2 q}{2} R_{12}. 
\]
Note that
\[
\frac{1}{N} U(\sigma, \sigma) = \frac{\beta^2}{4} \left( 1 - \frac{1}{N} \right) - \frac{\beta^2 q}{2}.
\]

Now plugging in the values of \( U(\sigma, \sigma), U(\sigma^1, \sigma^2) \) we have
\[
\varphi'(t) = \left( \frac{\beta^2}{4} - \frac{\beta^2 q}{2} \right) - \left( \frac{\beta^2}{4} \mathbb{E}(R^2_{12}) + \frac{\beta^2 q}{2} \mathbb{E}(R_{12}) \right) = -\frac{\beta^2}{4} \mathbb{E}((R_{12} - q)_t^2) + \frac{\beta^2}{4} (1 - q)^2.
\]

This gives, in particular,
\[
\varphi(1) \leq \varphi(0) + \frac{\beta^2 (1 - q)^2}{4} \quad \forall \ 0 \leq q \leq 1.
\]

Now note that
\[
\varphi(0) = \frac{1}{N} \mathbb{E} \log \left( \sum_{\sigma \in \{-1, +1\}^N} \prod_{i=1}^{N} \exp((\beta z_i \sqrt{q} + h) \sigma_i) \right)
= \frac{1}{N} \mathbb{E} \log \prod_{i=1}^{N} (\exp(\beta z_i \sqrt{q} + h) + \exp(-\beta z_i \sqrt{q} - h))
= \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \log (2 \cosh(\beta z_i \sqrt{q} + h)) = \log 2 + \mathbb{E} \log \cosh(\beta z \sqrt{q} + h)
\]

where \( z \sim N(0, 1) \). So for any \( 0 \leq q \leq 1 \), we have
\[
\mathbb{E} \left( \frac{\log Z_N}{N} \right) \leq \log 2 + \mathbb{E} \log \cosh(\beta Z \sqrt{q} + h) + \frac{\beta^2 (1 - q)^2}{4}.
\]

This inequality is called **Guerra’s inequality** and this holds for all \( \beta \geq 0, h \in \mathbb{R} \).

**Exercise 3** Prove that the R.H.S. of Guerra’s inequality is minimized when
\[
q = \mathbb{E} \tanh^2(\beta z \sqrt{q} + h).
\]