1 Some final remarks about the Curie-Weiss concentration

In the Curie Weiss model, if \( m(\sigma) = \frac{1}{n} \sum \sigma_i \), then
\[
P\left( |m - \tanh(\beta m)| \geq \frac{\beta}{n} + t \right) \leq \exp\left( -\frac{nt^2}{4(1+\beta)} \right).
\]
This implies that \( m - \tanh(\beta m) = O(1/\sqrt{n}) \) which is the optimal result when \( \beta \neq 1 \).

At \( \beta = 1 \), \( m - \tanh(\beta m) \propto n^{-3/4} \).

For \( \beta < 1 \) and all \( x \), \( |x - \tanh(\beta x)| \geq (1 - \beta)|x| \) and thus
\[
|m(\sigma)| \leq \frac{|m(\sigma) - \tanh(\beta m(\sigma))|}{1 - \beta} = O(1/\sqrt{n}).
\]

In particular \( P_\beta(|m(\sigma)| > \beta/n + t) \leq 2\exp\left( -\frac{n(1-\beta)^2t^2}{4(1+\beta)} \right) \).

When \( \beta = 1 \), \( P(|m(\sigma)| \geq t) \leq C\exp(-cnt^4) \) where \( C, c \) are not dependent on \( n \). This implies \( m = O(n^{-1/4}) \). Using Stein’s method one can further show that \( n^{1/4}m(\sigma) \) converges in distribution to \( Ce^{-x^4/12} \).

Exercise 1: Prove the above inequality using the exchangeable pair theorem.

Sketch: Use part (c) of the theorem. Use \( P(|m| \geq t) \leq \frac{E(m^2k)}{t^2k} \) and optimize over \( k \). Recall: \( f(\sigma) = m(\sigma) - 1/n \sum \tanh(\beta m(\sigma)) \), show that when \( \beta = 1 \), \( |f(\sigma) - f(\sigma')| \leq c\|m(\sigma)\|^2/n + c/n^2 \).

Also, \( |m(\sigma)|^3 \leq C|f(\sigma)| + C|m(\sigma)|/n \). Now combine.

2 KMT Strong Embedding

Theorem. (Komlós-Major-Tusnády) Suppose \( \epsilon_1, \epsilon_2, \ldots \) are i.i.d. with finite moment generating functions in a neighborhood of 0 with mean 0 and variance 1. Let \( S_n = \sum_{i=1}^n \epsilon_i \). We
can construct a version of \((S_k)_{k \geq 0}\) and a standard Brownian motion \(B\) on the same space such that for all \(n\), and \(t \geq 0\)

\[
P(\max_{1 \leq k \leq n} |S_k - B_k| \geq C \log n + t) \leq Ke^{-\lambda t}
\]

where \(C, K, \lambda\) depend only on the distribution of \(\epsilon_1\).

We will prove this result for the simple random walk using ideas from Stein’s method. We proceed in a series of lemmas.

**Lemma 1** Let \(n\) be a positive integer and suppose \(A\) is a continuous map from \(\mathbb{R}^n\) to the set of \(n \times n\) positive semidefinite matrices. Suppose the \(\|A\|\) is bounded by a \(b < \infty\). Then there exists a probability measure \(\mu\) such that if random variable \(X \sim \mu\) then for all \(\theta \in \mathbb{R}^n\),

\[
E \exp(\theta, X) \leq \exp(b\|\theta\|^2) \text{ and } E \langle X, \nabla f(X) \rangle = E \text{Tr}(A(X) \text{Hess} f(X))
\]

for all \(f \in C^2(\mathbb{R}^n)\) such that \(E |f(X)|^2, E \|\nabla f(X)\|^2, E |\text{Tr}(A(X) \text{Hess} f(X))| < \infty\).

**Proof:** Let \(K\) denote the set of all probability measures \(\mu\) on \(\mathbb{R}^n\) such that \(\int_{\mathbb{R}^n} x d\mu = 0\) and \(\int \exp(\theta, x) d\mu \leq \exp(b\|\theta\|^2)\) for all \(\theta \in \mathbb{R}^n\). By Skorokhod representation and Fatou’s lemma \(K\) is a nonempty, compact, and convex subset of the space \(V\) of finite signed measures on \(\mathbb{R}^n\).

Aside: \(K\) is closed, and compactness follows from tightness.

We now use the following:

**Schauder-Tychonoff Fixed Point Theorem:** A continuous map from a nonempty, convex, compact subset \(K\) of a locally convex topological space into \(K\) has a fixed point.

To be continued.