1. 3.1.7
Let \(\rho(d - \zeta)\) be an invariant loss function. and
\[
\delta - \zeta = X_11\{X_3 > 0\} + X_21\{X_3 \leq 0\} - \zeta
\]
\[
= (X_1 - \zeta)1\{X_3 > 0\} + (X_2 - \zeta)1\{X_3 \leq 0\}
\]
\[
\therefore R(\zeta, \delta) = E_\zeta[\rho(\delta - \zeta)]
\]
\[
= E_\zeta[\rho(X_1 - \zeta)1\{X_3 > 0\} + \rho(X_2 - \zeta)1\{X_3 \leq 0\}]
\]
\[
= E_0[\rho(X_1)1\{X_3 > -\zeta\} + \rho(X_2)1\{X_3 \leq -\zeta\}]
\]
\[
\therefore E_0[\rho(X_1)] = E_0[\rho(X_2)]
\]
\[
\therefore R(\zeta, \delta) = E_0[\rho(X_1)]
\]
So we got that \(R(\zeta, \delta)\) is constant for any invariant loss function.

But
\[
\delta + a = (X_1 + a)1\{X_3 > 0\} + (X_2 + a)1\{X_3 \leq 0\}
\]
\[
\neq (X_1 + a)1\{X_3 + a > 0\} + (X_2 + a)1\{X_3 + a \leq 0\}
\]
So \(\delta\) is not location equivariant.

2. 3.1.8
Suppose \(0 \leq \rho(t) \leq M\) for all \(t\) and \(M < \infty\). Further, suppose that \(\rho(t) \to M\) as \(t \to \pm \infty\) and that the density, \(f\), of \(X\) is continuous a.e. Then, let \(\phi(v) = E_0[\rho(X - v)]\). Then use Lebesgue’s Bounded Convergence theorem by noting \(|\rho(t)| = \rho(t) \leq M\) for all \(t\), and considering the measure space \((\mathbb{R}, B, \mu)\) where \(\mu\) is the Lebesgue measure. Then \(M\) is \(\mu\) integrable,
\[
\int_\mathbb{R} M d\mu(\omega) = \int_\mathbb{R} M d\mathcal{P}_X(x) = \int_\mathbb{R} M f(x) dx = M < \infty.
\]
such that by the BCT
\[
\lim_{v \to \infty} \phi(v) = \lim_{v \to \infty} \int_{-\infty}^{\infty} \rho(x - v) f(x) dx = \lim_{v \to \infty} \int_{-\infty}^{\infty} \rho(X(\omega) - v) d\mu(\omega)
\]
\[
= DCT \int_{-\infty}^{\infty} \lim_{v \to \infty} \rho(X(\omega) - v) d\mu(\omega) = \int_{-\infty}^{\infty} M d\mu(\omega)
\]
\[
= \int_{-\infty}^{\infty} M f(x) dx = M.
\]

It is clear that \(\lim_{v \to \infty} \phi(v) = M\) can be shown by an analogous calculation.
The function \(\phi(v)\) is continuous if \(\lim_{n \to \infty} v_n = v\) implies \(\lim_{n \to \infty} \phi(v_n) = \phi(\lim_{n \to \infty} v_n) = \phi(v)\). The proof of this is almost exactly as above. Suppose \(v_n \to v\),
\[
\lim_{n \to \infty} \phi(v_n) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \rho(x - v_n) f(x) dx = \int_{-\infty}^{\infty} \rho(u) f(u + v_n) du
\]
\[
= \int_{-\infty}^{\infty} \rho(u) f(u + v) du = \int_{-\infty}^{\infty} \rho(x - v) f(x) dx = \phi(v)
\]
the third equation is by the result given in the appendix 2 because $\rho$ is bounded and 
\[ \lim_{n \to \infty} f(x + v_n) = f(x + v) \text{ a.e. by the a.e. continuity of } f, \text{ so } f(x + v_n) \text{ weakly converge to } f(x+v). \]
Finally, we have that $\phi$ is continuous and that $\lim_{n \to \pm \infty} \phi(v) = M$. Therefore for all $\epsilon > 0$
there exists $a(\epsilon), b(\epsilon) \in \mathbb{R}$ such that $\phi(t) > M - \epsilon$ for all $t \notin [a(\epsilon), b(\epsilon)]$. Because $\phi$ is
continuous and $[a(\epsilon), b(\epsilon)]$ is compact, $\phi$ will attain its minimum over $[a(\epsilon), b(\epsilon)]$ for some
$t^* \in [a(\epsilon), b(\epsilon)]$. Thus if there exists $t_0$ such that $\phi(t_0) < M$ letting $\epsilon = M - \phi(t_0)$ will
ensure that $\phi$ attains its global minimum in $[a(\epsilon), b(\epsilon)]$. Therefore an MRE exists.

3. 3.1.10

Begin by noticing

\[ E[\rho(X - v)] = \int_{-\infty}^{\infty} \rho(x - v)f(x)dx \]
\[ = -\int_{-\infty}^{v} Axf(x)dx + \int_{v}^{\infty} Avf(x)dx + \int_{v}^{\infty} Bxf(x)dx - \int_{-\infty}^{v} Bvf(x)dx \]
\[ = AvF(v) - Bv[1 - F(v)] + \int_{v}^{\infty} Bxf(x)dx - \int_{-\infty}^{v} Axf(x)dx \]

such that because

\[ \frac{d}{dv} \left[ \int_{-\infty}^{v} Axf(x)dx \right] = Avf(v) \]

\[ \frac{d}{dv} E[\rho(X - v)] = [AF(v) + Avf(v) - B + Bvf(v) + BF(v)] - Bvf(v) - Avf(v) \]
\[ = F(v)[A + B] - B = 0 \]

implies $F(v) = B/(B + A)$, as desired. Next note that

\[ \frac{d^2}{dv^2} E[\rho(X - v)] = (A + B)f(v) \geq 0 \]

such that the extremum we have found is indeed a minimum.
Furthermore, since $A + B \geq 0$ and $F(v)$ is weakly increasing, it follows that the derivative
is weakly increasing in $v$. Hence, $E[\rho(X - v)]$ must be globally minimized at $F(v) = B/(B + A)$.

4. 3.1.11

Let $X_1, X_2, \ldots, X_n$ be iid $N(\xi, \sigma^2)$ where $\sigma$ is known. Then $\delta_0 = \bar{X}$ is a complete, sufficient
statistic for $\xi$, and moreover, that it is independent of the vector $Y$ from Theorem 3.1.10 by
Basu’s Theorem. Hence, we must minimize $E_0[\rho(\bar{X} - v)]$ when $v$ is constant (as in Example
3.1.16). Since $\bar{X} \sim N(\xi, \sigma^2/n)$ we can use our result from the previous exercise and use
(for $\xi = 0$, as in the proof of Theorem 3.1.10)

\[ F(v) = P(\bar{X} \leq v) = \Phi \left( \frac{v}{\sigma/\sqrt{n}} \right) = \frac{B}{A + B} \]

such that the minimizing value is

\[ v^* = \Phi^{-1} \left( \frac{B}{A + B} \right) \frac{\sigma}{\sqrt{n}}. \]
Finally, the MRE is given by
\[ \bar{X} - v^* = \bar{X} - \Phi^{-1} \left( \frac{B}{A + B} \right) \frac{\sigma}{\sqrt{n}}. \]

It should also be clear that no MRE exists if \( A = 0 \) or \( B = 0 \), since in this case smaller and smaller risk can be attained as \( v^* \to \pm \infty \).

5. 3.1.14
Suppose \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) have joint density
\[ f(x_1 - \xi, \ldots, x_m - \xi; y_1 - \eta, \ldots, y_n - \eta) \tag{1} \]
and consider the problem of estimating \( \Delta = \xi - \eta \). An estimator will be called “good” if it satisfies
\[ \delta(x + a, y + b) = \delta(x, y) + (b - a). \]

The following results are analogous to those provided in the text:

**Theorem 1.4** Let \( X, Y \) be distributed as in (1), and \( \delta \) be equivariant with loss function \( L((\xi, \eta), d) = \rho(d - \Delta) \). Then the bias, risk and variance of \( \delta \) are all constant.

**Proof.** As in the book, it suffices to illustrate the case of bias because risk and variance are shown by the same process,
\[ b(\Delta) = E_{\xi, \eta}[\delta(X, Y)] - \Delta = E_0[\delta(X + \xi, Y + \eta) - \Delta] = E_0[\delta(X, Y)] \]
which does not depend on \( \xi \) or \( \eta \).

**Lemma 1.6** If \( \delta_0 \) is equivariant then \( \delta \) is also equivariant iff \( \delta = \delta_0 + u \) where \( u(x + a, y + b) = u(x, y) \) for all \( x, y \) and \( a, b \).

**Proof.** Assume that \( \delta = \delta_0 + u \) where \( u(x + a, y + b) = u(x, y) \) for all \( x, y \) and \( a, b \). Then
\[ \delta(x + a, y + b) = \delta_0(x + a, y + b) + u(x + a, y + b) = \delta(x, y) + (b - a) \]
such that \( \delta \) is equivariant. In the other direction, assume that \( \delta \) is equivariant and let \( u(x, y) = \delta(x, y) - \delta_0(x, y) \) such that
\[ u(x + a, y + b) = \delta(x + a, y + b) - \delta_0(x + a, y + b) = \delta(x, y) - \delta_0(x, y) = u(x, y) \]
as desired.

**Lemma 1.7** A function \( u \) satisfies \( u(x + a, y + b) = u(x, y) \) iff it is a function of the differences \( z_i = x_i - x_m \) for \( i = 1, \ldots, m \) and \( w_i = y_i - y_n \) for \( i = 1, \ldots, n \).

**Proof.** If \( u \) is a function of \( x \) and \( w \) then it is clear that \( u(x + a, y + b) = u(x, y) \). In the other direction, the result follows from letting \( a = -x_m \) and \( b = -y_n \).
\[ u(x, y) = u(x_1 + a, \ldots, x_m + a; y_1 + b, \ldots, y_n + b) = u(z_1, \ldots, z_{m-1}, 0; w_1, \ldots, w_{n-1}, 0) \]
as desired.

**Theorem 1.8** If \( \delta_0 \) is equivariant, then a necessary and sufficient condition for \( \delta \) to be good is that there exists a function \( v \) of \( n - 1 + m - 1 = n + m - 2 \) arguments such that
\[ \delta(x, y) = \delta_0(x, y) - v(z, w). \]
Proof. Lemmas 1.6 and 1.7.

Lemma 1.10 Let $X, Y$ be distributed according to (1), and let $Z$ and $W$ denote the differences, as introduced above. Then suppose there exists a good estimator of $\Delta$ with finite risk. Further, assume for each $(z, w)$ there exists a number $v(z, w) = v^*(z, w)$ such that $v^*$ minimizes

$$E_0[\rho(\delta_0(X, Y) - v(z, w)) | z, w].$$

Then an MRE exists.

Proof. By Theorem 1.8 we want minimize

$$E_{\xi, \eta}[\rho(\delta_0(X, Y) - v(z, w) - \Delta)]$$

but it is sufficient to consider $\xi, \eta = 0$ because the risk does not depend on $\xi$ or $\eta$. Then we seek to minimize

$$E_0[\rho(\delta_0(X, Y) - v(z, w))] = \int E_0[\rho(\delta_0(X, Y) - v(z, w)) | z, w] dP_0(z, w)$$

for which is sufficient to minimize the integrated $E_0[\rho(\delta_0(X, Y) - v(z, w)) | z, w]$, which has minimum by assumption. Thus, an MRE estimator of $\Delta$ exists.

Corollary 1.11 Suppose $\rho$ is convex and not monotone. Then $E_0[\rho(\delta_0(X, Y) - v(z, w)) | z, w]$ has minimum as a direct result of Theorem 1.7.15, such that an MRE of $\Delta$ exists by Theorem 1.10.

Corollary 1.12 If $\rho$ is squared error loss then $v^*(z, w) = E_0[\delta_0(X, Y) | z, w]$ by example 1.7.17. Next, if $\rho$ is absolute error loss then $V^*$ is any median of $\delta_0(X, Y)$ under the conditional distribution of $(X, Y)$ given $z, w$ by example 1.7.18.

Corollary 1.14 Example 1.13 implies that if $m = n = 1$, and if $(Y - X)$ has finite risk then $v^*$ is any value minimizing

$$E_0[\rho((Y - X) - v)].$$

Suppose that $0 \leq \rho(t) \leq M$ and that $\rho(t) \to M$ as $t \to \pm \infty$ and that the density of $Y - X$ is a.e. continuous. Then an MRE estimator of $\Delta$ exists. The proof of this corollary is exactly analogous to exercise 1.8 in this homework set.

Theorem 1.17 Let $\mathcal{F}$ be the class of all univariate distributions $F$ that have a density and fixed finite variance $\sigma^2 = 1$. Let $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ be iid with density $f(x_1 - \xi, \ldots, x_m - \xi; y_1 - \eta, \ldots, y_n - \eta)$ and let $r_n(F)$ be the risk of the MRE estimator of $\xi - \eta$ with squared error loss. Then, $r_n(F)$ takes on its maximum value over $\mathcal{F}$ when $F$ is normal.

Proof. The MRE estimator for $\xi$ and $\eta$ in the normal case are $\bar{X}$ and $\bar{Y}$ respectively. Then by the original proof,

$$E[(\bar{Y} - \bar{X} - \Delta)^2] = E[(\bar{Y} - \eta) - (\bar{X} - \xi))^2]
= E[(\bar{X} - \xi)^2] + E[(\bar{Y} - \eta)^2] - 2E[(\bar{Y} - \eta)(\bar{X} - \xi)]
= E[(\bar{X} - \xi)^2] + E[(\bar{Y} - \eta)^2] = \frac{1}{n} + \frac{1}{m}.$$ 

Since the risk did not depend on $F$, the MRE estimator of any other $F$ must have risk $\leq 1/n + 1/m$, completing the proof.
6. 3.1.16
If the $X_i$'s and $Y_i$'s are independently normally distributed with known variances, we know from Example 1.6.20 that the statistics $\tilde{X}_i = X_i - X_n$ are ancillary for $\xi$ and the statistics $\tilde{Y}_i = Y_i - Y_n$ are ancillary for $\eta$. Clearly, $\tilde{X}_i$ must be ancillary for $\eta$ since the distribution of $X$ does not depend on $\eta$, and similarly $\tilde{Y}_i$ must be ancillary for $\xi$. Thus, $(\tilde{X}, \tilde{Y})$ is ancillary for $(\xi, \eta)$. Since $(\bar{X}, \bar{Y})$ is complete sufficient for $(\xi, \eta)$ by Theorem 1.6.22, it follows from Basu’s Theorem that any function of $(\bar{X}, \bar{Y})$ is independent of $(\tilde{X}, \tilde{Y})$, and in particular so is $\delta_0(X, Y) = \bar{Y} - \bar{X}$. Since $\delta_0$ has the property in Exercise 3.1.13, it qualifies in our search for a minimum risk estimator. By Theorem 1.10, if $\delta_0$ has finite risk, then the MRE estimator can be found by minimizing $E_0\{\rho[\delta_0(X, Y) - v(x, y)]\} = E_0\{\rho[\delta_0(X, Y) - v]\}$ by independence. Clearly, $\delta_0$ will be MRE as long as this expression is minimized at $v^* = 0$. Since $\delta_0$ is a linear combination of Gaussians, it is itself Gaussian with mean zero under $\xi = \eta = 0$.

Therefore, by Corollary 1.7.19, if $\rho$ is convex and even, then $\bar{Y} - \bar{X}$ will be MRE. Similarly, by Exercise 1.7.23, the same is true if $\rho$ is even and non-decreasing on $(0, \infty)$ (in fact, the conditions in this sentence and the previous one imply each other).

7. 3.1.22
By Lemma 3.1.6, any equivariant estimator $\delta$ can be written as $\delta = \delta_0 + u$, where $\delta_0$ is any equivariant estimator and $u$ is any location invariant function.

Suppose that $\delta_0$ is MRE, by lemma 3.1.23, it is unbiased under square error loss function. Also by Lemma 3.1.23, if $\delta$ is biased with constant bias $b$, then $\delta - b$ is equivariant, unbiased and has smaller risk than $\delta$. So that $\delta_0$ is MRE among all the unbiased equivariant estimators is equivalent to $\delta_0$ is MRE. Then by the similar proof of Theorem 2.1.7, we can get the result.