1. (a) Let us use results that we know about the gambler’s ruin problem. Indeed, $M_\tau = 0$ iff the random walk reaches $-a$ before 1. Translated in the gambler’s game, this is the event of starting with a fortune of 1 and winning against an opponent having a fortune $a$. So,

$$P(M_\tau = 0) = \frac{1}{a+1}$$

(b) On $M_\tau \geq 2$, the process reaches 1 before $-a$; from 1 starts refreshed and reaches 2 before $-a+1$. Those “two” events have same probability $P(M_\tau \geq 1)$, and therefore

$$P(M_\tau \geq 2) = P(M_\tau \geq 1)^2$$

In the same way,

$$P(M_\tau \geq k) = P(M_\tau \geq 1)^k$$

Now,

$$P(M_\tau \geq 1) = 1 - P(M_\tau = 0) = \frac{a}{a+1}$$

Hence, $M_\tau$ has a geometric distribution with parameter $1/(a+1)$.

(c) Define $B_n(t) = S_{[nt]}/\sqrt{n}$, and the corresponding $M_n(t)$ and $Y_n(t)$. Let

$$\tau_n = \min\{t \geq 0 : Y_n(t) \geq a\}$$

The invariance principle says that, as $n \to \infty$,

$$P(M_n(\tau_n) > u) \to P(M(\tau) > u)$$
We want to use the previous results:

\[ M_k = \sqrt{n}M_n(k/n) \]

and so,

\[ P(M_n(\tau_n) > u) = P(M_{n\tau_n} \geq \lceil \sqrt{n}u \rceil + 1) = \left( \frac{\sqrt{n}a}{\sqrt{n}a + 1} \right)^{\lceil \sqrt{n}u \rceil + 1} \]

with the last term on the right tending to \( \exp(-u/a) \) as \( n \to \infty \). Hence, \( M(\tau) \) has exponential distribution with mean \( a \).

2. Fix \( t > 0 \) and call

\[ R^t_c = \{ B(s) \neq 0 \forall s \in (t, t + c) \} \]

We have \( R^t_b \subset R^t_a \), since \( a < b \), and so

\[ P(R^t_b | R^t_a) = \frac{P(R^t_b)}{P(R^t_a)} \]

Also,

\[ P(R^t_c) = 1 - \frac{2}{\pi} \arctan \sqrt{c/t} \]

Therefore,

\[ P(R^t_b | R^t_a) = \frac{1 - \frac{2}{\pi} \arctan \sqrt{b/t}}{1 - \frac{2}{\pi} \arctan \sqrt{a/t}} \]

3. We know that

\[ \pi/2 - \arctan(x) = \arctan(1/x) \]

with \( \arctan(x) \propto x \) at zero. Therefore, as \( t \to 0 \),

\[ \pi/2 \ P(R^t_c) = \arctan \sqrt{t/c} \propto \sqrt{t/c} \]

As a consequence, as \( t \to 0 \),

\[ P(R^t_b | R^t_a) \propto \frac{\sqrt{t/b}}{\sqrt{t/a}} = \frac{a}{b} \]
4. Starting from page 493,

\[ P(M(t) > x) = 2P(B(t) > x) = P(|B(t)| > x) \]

the last one being by symmetry of Brownian motion. Therefore, \( M(t) \)
and \(|B(t)|\) have same law. Differentiate (2.2) page 493 with respect to \( x \) to find \( f_{M(t)} \).

\( M(t) \) has the law of \( \sqrt{t}|Z| \), with \(|Z|\) standard normal. We know that \( E[|Z|] = \sqrt{2/\pi} \), so \( E[M(t)] = \sqrt{2t/\pi} \).

By definition \( M(s) \leq M(t) \) for \( s < t \), while \(|B(s)| > |B(t)|\) with positive probability. Hence, the answer is NO.

5. We need to assume \( x < z \). Call \( A = \{M(t) \geq z, B(t) \leq x\} \). On \( A \) the process reaches \( z \). From there, at time \( t \), the process is under \( x \). This corresponds to change of level of \( x - z \). This has same probability as a change of level of \( z - x \), which is to say that \( B(t) \) reaches \( 2z - x \). That new path includes the first piece - the reaching of \( z \). Hence the result.

Remark : We reflect the process from the first time it reaches \( z \), with respect to the level \( z \).

Note : Finding the density is just differentiation. However, it should include the restrictions \( x \leq z \) and \( z \geq 0 \).

6. Given the previous result, this is just the change of variable \( (z, x) \rightarrow (z, z - x) \). The Jacobian is indeed 1. As before, there are restrictions \( y, z \geq 0 \).

7. In other words, we want the marginal of \( Y \). For that we integrate the previous expression with respect to \( z \). So fix \( y \), and change variable \( u = (z + y)/\sqrt{t} \), to get

\[
\int_0^\infty \frac{z + y}{t} \frac{2}{\sqrt{t}} \phi \left( \frac{z + y}{\sqrt{t}} \right) dz = \frac{2}{\sqrt{t}} \int_{y/\sqrt{t}}^\infty u \phi(u) du
\]

But \( d/du \phi(u) = -u \phi(u) \), so we get the result.