Statistics 200, Homework 5
Due Tuesday, February 24, 2009

The first four problems are from the text, Chapter 8: 16, 52, 59, 68.

5. Suppose $X_1, \ldots, X_n$ are i.i.d. with the following mixture density. With probability $p$, each $X_i$ is $N(\mu, 1)$ and with probability $1 - p$, $X_i$ is $N(\mu, \tau^2)$, where $\tau > 1$. As a function of $p$, $\mu$ and $\tau^2$, compute $E(X_i)$ and $Var(X_i)$.

6. (continuation of 5) For $n$ large, compare the variances of the sample mean $\bar{X}_n$ and the sample median $M_n$. Compute the ratio of asymptotic variances if you know that $p = .9$ and $\tau = 5$. If you base a confidence interval for $\mu$ on $\bar{X}_n$, about how large a sample size would you need so that a 95 percent confidence interval has length 0.1? Answer the same question if the confidence interval is based on $M_n$.

7. Suppose $X_1, \ldots, X_n$ are i.i.d. with the Cauchy distribution, i.e. the density is

$$f(x, \theta) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}.$$

Let $\tilde{\theta}_n$ be the sample median. What is the limiting distribution of $n^{1/2}(\tilde{\theta}_n - \theta)$? Use this limiting distribution to approximate the chance that $P\{|\tilde{\theta}_n - \theta| \leq 1/5\}$ in the case $n = 101$. Compare this probability to the one where $\tilde{\theta}_n$ is replaced by an efficient estimator $\hat{\theta}_n$ which satisfies $n^{1/2}(\hat{\theta}_n - \theta)$ converges in distribution to $N(0, I^{-1}(\theta))$. Hint: For the Cauchy model, $I(\theta) = 1/2$. 

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