4.6 \( X \) is a continuous random variable with probability density function \( f(x) = 2x, 0 \leq x \leq 1 \).

(a) The expectation of \( X \) is found by integrating \( xf(x) \) over the interval \([0, 1]\) on which the density is non-zero.

\[
E[X] = \int_0^1 xf(x) \, dx = \int_0^1 2x^2 \, dx = \frac{2}{3} x^3 \bigg|_0^1 = \frac{2}{3}
\]

(b) For \( Y = X^2 \), we first calculate the density of \( Y \) by finding the CDF then differentiating. Note that \( 0 \leq X \leq 1 \) implies \( 0 \leq Y \leq 1 \)

\[
P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})
= P(X \leq \sqrt{y}) \quad \text{(since } X \text{ must be positive)}
= \int_0^{\sqrt{y}} 2x \, dx = x^2 \bigg|_0^{\sqrt{y}} = y, 0 \leq y \leq 1
\]

\[
\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(Y) = 1, 0 \leq y \leq 1
\]

Then to find the expectation of \( Y \) we integrate \( yf_Y(y) \) over the integral \([0, 1]\) on which the density
of $Y$ is non-zero.

$$E[Y] = \int_0^1 y f(y) \, dy$$
$$= \int_0^1 y \, dy$$
$$= \frac{1}{2}y^2 \bigg|_0^1$$
$$= \frac{1}{2}$$

(c) By Theorem A in Section 4.1.1, we have $E[g(X)] = \int g(x) f(x) \, dx$, so for $Y = g(X) = X^2$ we have:

$$E[Y] = \int_0^1 x^2 f(x) \, dx$$
$$= \int_0^1 2x^3 \, dx$$
$$= \frac{2}{4}x^4 \bigg|_0^1$$
$$= \frac{1}{2}$$

(d) The definition of the variance of a random variable $X$ is $\text{Var}(X) = E[(X - E[X])^2]$, so using Theorem A of Section 4.1.1 again with $g(X) = (X - E[X])^2$, we find that $\text{Var}(X)$ is:

$$\text{Var}(X) = E[(X - E[X])^2]$$
$$= \int_0^1 \left(x - \frac{2}{3}\right)^2 2x \, dx$$
$$= \int_0^1 \left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) 2x \, dx$$
$$= \int_0^1 \left(2x^3 - \frac{8}{3}x^2 + \frac{8}{9}x\right) \, dx$$
$$= \left(\frac{2}{4}x^4 - \frac{8}{9}x^3 + \frac{4}{9}x^2\right) \bigg|_0^1$$
$$= \frac{1}{2} - \frac{4}{9}$$
$$= \frac{1}{18}$$

Theorem B of Section 4.2 gives an alternate expression for the variance of $X$ as $\text{Var}(X) = E[X^2] - (E[X])^2$. We know $E[X^2] = E[Y] = \frac{1}{2}$ from parts (b) and (c), and we know $E[X] = \frac{2}{3}$ from
part(a), so combining these results we have:

\[
\text{Var}(X) = E[X^2] - (E[X])^2 \\
= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\
= \frac{1}{2} - \frac{4}{9} \\
= \frac{1}{18}
\]

4.28 Let \( X \) be the number of aircraft hit by gunners, so we want to find \( E[X] \). Define \( X_i \) to be an indicator variable indicating the event that the \( i^{th} \) aircraft is hit, so

\[
X_i = \begin{cases} 
1 & \text{if aircraft } i \text{ is hit} \\
0 & \text{otherwise}
\end{cases}
\]

Then \( X = \sum_{i=1}^{n} X_i \), and by linearity of expectation we have \( E[X] = \sum_{i=1}^{n} E[X_i] \). Now we must find \( E[X_i] \), which will be the same for all \( i = 1, \ldots, n \) because the aircraft are exchangeable.

\[
E[X_i] = E[X_i] = 1 \times P(X_i = 1) + 0 \times P(X_i = 0) \\
= P(X_i = 1) \\
= 1 - P(X_i = 0)
\]

Now define \( Z_{ij} \) to be the indicator that gunner \( j \) hits plane \( i \), so

\[
Z_{ij} = \begin{cases} 
1 & \text{if gunner } j \text{ hits aircraft } i \\
0 & \text{otherwise}
\end{cases}
\]

Clearly, the only way that plane \( i \) is not hit is if none of gunners \( j \) for \( j = 1, \ldots, m \) hit plane \( i \), so \( \{X_i = 0\} \iff \{Z_{ij} = 0 \text{ for all } j = 1, \ldots, m\} \). We can easily calculate \( P(Z_{ij} = 1) \) using the Law of Total Probability, letting \( B_i \) denote the event that gunner \( j \) selects aircraft \( i \) as a target.

\[
P(Z_{ij} = 1) = P(Z_{ij} = 1 \mid B_i) P(B_i) + P(Z_{ij} = 1 \mid B_i^C) P(B_i^C) \\
= p \frac{1}{n} + 0 \left(1 - \frac{1}{n}\right) \\
= p \frac{1}{n} \\
\Rightarrow P(Z_{ij} = 0) = 1 - P(Z_{ij} = 1) \\
= 1 - p \frac{1}{n}
\]

Then since the gunners choose and hit their targets independently, we have:

\[
P(X_i = 0) = P(Z_{ij} = 0, j = 1, \ldots m) \\
= P(Z_{i1} = 0) P(Z_{i2} = 0) \ldots P(Z_{im} = 0) \\
= \left(1 - \frac{p}{n}\right)^m
\]
So $E[X_i] = 1 - P(X_i = 0) = 1 - \left(1 - \frac{p}{n}\right)^m$, and therefore, substituting back into the original expression for $E[X]$,

$$E[X] = \sum_{i=1}^n E[X_i] = n \left[1 - \left(1 - \frac{p}{n}\right)^m\right] = n - n \left(1 - \frac{p}{n}\right)^m$$

[NOTE: This problem could also be solved by calculating $P(X_i = 1)$ directly, conditioning on the number of gunners who target plane $i$. This approach will result in a more complicated sum, but it is equally correct and ultimately reduces to the same answer.]

4.42 For $X \sim \text{Exp}(\lambda)$, we have $E[X] = \frac{1}{\lambda}$, $\text{Var}(X) = \frac{1}{\lambda^2}$, and the CDF is given by $F_X(x) = 1 - e^{-\lambda x}$. We are told that the standard deviation of $X$ is $\sigma$, which means that

$$\sigma^2 = \text{Var}(X) = \frac{1}{\lambda^2} \quad \Rightarrow \quad \sigma = \frac{1}{\lambda}$$

Then we compute $P \left(||X - E[X]| > k\sigma\right)$, substituting $E[X] = \sigma$ as follows:

$$P \left(||X - E[X]| > k\sigma\right) = P \left(|X - \sigma| > k\sigma\right)$$
$$= P \left(X - \sigma > k\sigma\right) + P \left((-X - \sigma) > k\sigma\right)$$
$$= P \left(X > (k + 1)\sigma\right) + P \left(-X > (k - 1)\sigma\right)$$
$$= P \left(X > (k + 1)\sigma\right) + P \left(X < (k - 1)\sigma\right)$$
$$= 1 - P \left(X \leq (k + 1)\sigma\right)$$
$$= 1 - \left(1 - e^{-\lambda(k+1)\sigma}\right)$$
$$= e^{-\lambda(k+1)^2\frac{1}{\lambda}}$$
$$= e^{-(k+1)}$$

Using Chebyshev’s inequality, we have $P \left(||X - E[X]| > t\right) \leq \frac{\sigma^2}{t^2}$. Substituting in $E[X] = \sigma$ and letting $t = k\sigma$, we find:

$$P \left(|X - \sigma| > k\sigma\right) \leq \frac{\sigma^2}{(k\sigma)^2}$$
$$= \frac{1}{k^2}$$

The values of the exact probability and the Chebyshev bound are given below.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Exact Probability</th>
<th>Chebyshev Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0498</td>
<td>0.2500</td>
</tr>
<tr>
<td>3</td>
<td>0.0183</td>
<td>0.1111</td>
</tr>
<tr>
<td>4</td>
<td>0.0067</td>
<td>0.0625</td>
</tr>
</tbody>
</table>
4.52 (a) The first security is the better choice because it has a higher expected return and a lower risk.

(b) The expected return is given by $E[R(\pi)]$ where $\pi = 0.5$:

\[
E[R(\pi)] = \pi \mu_1 + (1 - \pi)\mu_2
\]

\[
E[R(0.5)] = 0.5(1) + 0.5(0.8) = 0.9
\]

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.5$:

\[
\text{Var}(R(\pi)) = \pi^2 \sigma_1^2 + 2\pi(1 - \pi)\rho\sigma_1\sigma_2^2 + (1 - \pi)^2\sigma_2^2
\]

\[
\text{Var}(R(0.5)) = 0.5^2 (0.1^2) + 2 (0.5) (0.5) (-0.8) (0.1) (0.12) + 0.5^2 (0.12^2)
\]

\[
= 0.0013
\]

\[
\Rightarrow \sqrt{\text{Var}(R(0.5))} = 0.0361
\]

(c) The expected return is given by $E[R(\pi)]$ where $\pi = 0.8$:

\[
E[R(\pi)] = \pi \mu_1 + (1 - \pi)\mu_2
\]

\[
E[R(0.8)] = 0.8(1) + 0.2(0.8) = 0.96
\]

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.8$:

\[
\text{Var}(R(\pi)) = \pi^2 \sigma_1^2 + 2\pi(1 - \pi)\rho\sigma_1\sigma_2^2 + (1 - \pi)^2\sigma_2^2
\]

\[
\text{Var}(R(0.8)) = 0.8^2 (0.1^2) + 2 (0.8) (0.2) (-0.8) (0.1) (0.12) + 0.2^2 (0.12^2)
\]

\[
= 0.0039
\]

\[
\Rightarrow \sqrt{\text{Var}(R(0.5))} = 0.0625
\]

(d) The red line of Figure 1 displays the plot of $(\mu(\pi), \sigma(\pi))$ as $\pi$ varies from 0 to 1.

(e) The expected return is unchanged for different values of $\rho$, and is given by $E[R(\pi)]$ where $\pi = 0.5$:

\[
E[R(0.5)] = 0.9
\]

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.5$. When $\rho = 0.1$, we find:

\[
\text{Var}(R(\pi)) = \pi^2 \sigma_1^2 + 2\pi(1 - \pi)\rho\sigma_1\sigma_2^2 + (1 - \pi)^2\sigma_2^2
\]

\[
\text{Var}(R(0.5)) = 0.5^2 (0.1^2) + 2 (0.5) (0.5) (0.1) (0.12) + 0.5^2 (0.12^2)
\]

\[
= 0.0067
\]

\[
\Rightarrow \sqrt{\text{Var}(R(0.5))} = 0.0819
\]

The expected return is unchanged for different values of $\rho$, and is given by $E[R(\pi)]$ where $\pi = 0.8$:

\[
E[R(0.8)] = 0.96
\]

The risk is the standard deviation of $R(\pi)$, i.e., $\sqrt{\text{Var}(R(\pi))}$ where $\pi = 0.8$. When $\rho = 0.1$ we find:

\[
\text{Var}(R(\pi)) = \pi^2 \sigma_1^2 + 2\pi(1 - \pi)\rho\sigma_1\sigma_2^2 + (1 - \pi)^2\sigma_2^2
\]

\[
\text{Var}(R(0.8)) = 0.8^2 (0.1^2) + 2 (0.8) (0.2) (0.1) (0.12) + 0.2^2 (0.12^2)
\]

\[
= 0.0074
\]

\[
\Rightarrow \sqrt{\text{Var}(R(0.8))} = 0.0858
\]
Figure 1: Plot of \((\mu(\pi), \sigma(\pi))\) as \(\pi\) varies from 0 to 1. The red line displays the curve when \(\rho = -0.8\), and the green line displays the curve when \(\rho = 0.1\). The black points are \((\mu(\pi), \sigma(\pi))\) for the value of \(\pi\) that is printed below or above the respective point.

The green line of Figure 1 displays the plot of \((\mu(\pi), \sigma(\pi))\) when \(\rho = 0.1\) as \(\pi\) varies from 0 to 1.

4.58 We are given \(X_1 = f(x) + \varepsilon_1\) and \(X_2 = f(x + h) + \varepsilon_2\) where \(\varepsilon_1\) and \(\varepsilon_2\) are independent with mean 0 and variance \(\sigma^2\). The derivative \(f'(x)\) is estimated by the random variable \(Z = \frac{X_2 - X_1}{h}\).

(a) The expectation of \(Z\) is found using the linearity of expectation property:

\[
E[Z] = E\left[ \frac{X_2 - X_1}{h} \right]
= \frac{1}{h} (E[X_2] - E[X_1])
= f(x + h) - f(x)
\]

And the variance of \(Z\) is found using the properties of a variance of a sum of independent random
variables given on page 140 of the text:

\[
\text{Var}(Z) = \text{Var}\left(\frac{X_2 - X_1}{h}\right) = \frac{1}{h^2} (\text{Var}(X_2) + \text{Var}(X_1)) = \frac{2\sigma^2}{h^2}
\]

Clearly, \(\text{Var}(Z) \to \infty\) as \(h \to 0\), so choosing a very small value of \(h\) will cause the variance to grow very large.

(b) The mean squared error (\(MSE\)) of an estimator \(X\) for a quantity \(x_0\) is given by

\[
MSE(X) = E[(X - x_0)^2]
\]

Therefore, the mean squared error of \(Z\) as an estimator of \(f'(x)\) is:

\[
\]

In order to proceed, we would need to know \(f'(x)\), which we do not know. However, using a Taylor series expansion, we can derive an approximation. The Taylor series expansion of \(f(x + h)\) about \(x\) gives:

\[
f(x + h) = f(x) + f'(x)(x + h - x) + \ldots
\]

\[
\Rightarrow f(x + h) \approx f(x) + f'(x)(h)
\]

\[
\Rightarrow f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

\[
\Rightarrow f'(x) \approx E[Z]
\]

Now plugging this approximation of \(f'(x)\) back into our expression for the mean squared error of \(Z\), we find:

\[
MSE \approx E[Z^2] - 2E[Z]E[Z] + (E[Z])^2 = E[Z^2] - (E[Z])^2 = \text{Var}(Z) = \frac{2\sigma^2}{h^2} \quad (MSE^*)
\]

Thus \(MSE^* \to 0\) as \(h \to \infty\), so we cannot find a value of \(h\) to minimize this approximation \(MSE^*\) of the mean squared error.

(c) Now we are given three points, which we will express as

\[
X_1 = f(x) + \varepsilon_1
\]

\[
X_2 = f(x + h) + \varepsilon_2
\]

\[
X_3 = f(x + h + k) + \varepsilon_3
\]

again assuming that \(\varepsilon_1, \varepsilon_2, \text{ and } \varepsilon_3\) are independent with mean 0 and common variance \(\sigma^2\). Following the example of the first part of this problem, we define

\[
Z_1 = \frac{X_2 - X_1}{h}
\]

\[
Z_2 = \frac{X_3 - X_2}{k}
\]
so $Z_1$ is an estimate of $g(x) = f'(x)$ and $Z_2$ is an estimate of $g(x+h) = f'(x+h)$. Again, we can analogously define

$$Y = \frac{Z_2 - Z_1}{h} = \frac{1}{h} \left( \frac{X_3 - X_2}{k} - \frac{X_2 - X_1}{h} \right) = \frac{1}{hk} X_3 - \left( \frac{1}{hk} + \frac{1}{h^2} k \right) X_2 + \frac{1}{h^2} X_1$$

which, by the same reasoning used in the set-up of the problem, is an estimate of $g'(x) = f''(x)$. Then we can compute the expectation and variance of $Y$:

$$E[Y] = E \left[ \frac{1}{hk} X_3 - \left( \frac{1}{hk} + \frac{1}{h^2} k \right) X_2 + \frac{1}{h^2} X_1 \right] = \frac{1}{hk} E[X_3] - \left( \frac{1}{hk} + \frac{1}{h^2} \right) E[X_2] + \frac{1}{h^2} E[X_1] = \frac{1}{hk} f(x + h + k) - \left( \frac{1}{hk} + \frac{1}{h^2} \right) f(x + h) + \frac{1}{h^2} f(x)$$

$$\text{Var}(Y) = \text{Var} \left( \frac{1}{hk} X_3 - \left( \frac{1}{hk} + \frac{1}{h^2} k \right) X_2 + \frac{1}{h^2} X_1 \right) = \frac{1}{(hk)^2} \text{Var}(X_3) + \left( \frac{1}{hk} + \frac{1}{h^2} \right)^2 \text{Var}(X_2) + \frac{1}{h^2} \text{Var}(X_1) = \sigma^2 \left( \frac{1}{h^2 k^2} + \frac{1}{h^2 k^2} + \frac{2}{h^4 k} + \frac{1}{h^4} \right) = 2 \sigma^2 \left( \frac{1}{h^2 k} + \frac{1}{h^4} \right)$$

4.102 We are given $\Theta = g(X, Y) = \tan^{-1} \left( \frac{X}{Y} \right)$, where $X$ has mean $x_0$ and variance $\sigma^2$, and $Y$ has mean $y_0$ and variance $\sigma^2$. Using the approximations given on page 165 of the text, we have:

$$E[g(X,Y)] \approx g(\mu_X, \mu_Y) + \frac{1}{2} \sigma_X^2 \frac{\partial g(\mu_X, \mu_Y)}{\partial x^2} + \frac{1}{2} \sigma_Y^2 \frac{\partial g(\mu_X, \mu_Y)}{\partial y^2} + \sigma_{XY} \frac{\partial g(\mu_X, \mu_Y)}{\partial x \partial y}$$

$$\text{Var}(g(X,Y)) \approx \sigma_X^2 \left( \frac{\partial g(\mu_X, \mu_Y)}{\partial x} \right)^2 + \sigma_Y^2 \left( \frac{\partial g(\mu_X, \mu_Y)}{\partial y} \right)^2 + 2\sigma_{XY} \left( \frac{\partial g(\mu_X, \mu_Y)}{\partial x} \right) \left( \frac{\partial g(\mu_X, \mu_Y)}{\partial y} \right)$$

In this problem, we are told that $X$ and $Y$ are independent, so the terms with $\sigma_{XY}$ disappear. We
must now calculate all of the remaining partial derivatives:

\[
\frac{\partial}{\partial x} g(x, y) = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \left(-\frac{y}{x^2}\right)
= \frac{-y}{x^2 + y^2}
\]

\[
\frac{\partial^2}{\partial x^2} g(x, y) = (-y)(-1) \left(x^2 + y^2\right)^{-2} (2x)
= \frac{2xy}{(x^2 + y^2)^2}
\]

\[
\frac{\partial}{\partial y} g(x, y) = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \left(\frac{1}{x}\right)
= \frac{1}{x + \frac{x}{y}}
= \frac{x}{x^2 + y^2}
\]

\[
\frac{\partial^2}{\partial y^2} g(x, y) = (x)(-1) \left(x^2 + y^2\right)^{-2} (2y)
= \frac{-2xy}{(x^2 + y^2)^2}
\]

Plugging these results into the approximation expressions, we find:

\[
E[\Theta] \approx \tan^{-1} \left(\frac{y_0}{x_0}\right) + \frac{1}{2} \sigma^2 \frac{2x_0y_0}{(x_0^2 + y_0^2)^2} + \frac{1}{2} \sigma^2 \frac{-2x_0y_0}{(x_0^2 + y_0^2)^2}
= \tan^{-1} \left(\frac{y_0}{x_0}\right)
\]

\[
\text{Var}(\Theta) \approx \sigma^2 \left(\frac{-y_0}{x_0^2 + y_0^2}\right)^2 + \sigma^2 \left(\frac{x_0}{x_0^2 + y_0^2}\right)^2
= \sigma^2 \left(\frac{y_0^2}{(x_0^2 + y_0^2)^2} + \frac{x_0^2}{(x_0^2 + y_0^2)^2}\right)
= \frac{\sigma^2}{x_0^2 + y_0^2}
\]

5.4 For \(N \sim \text{Poisson}(\lambda)\), we have \(E[N] = \text{Var}(N) = \lambda\). Here we are given that \(E[N] = 100\), so \(\lambda = 100\). To use the normal approximation to the Poisson, we must first standardize the Poisson random variable.
Let

\[ Z = \frac{N - E[N]}{\sqrt{Var(N)}} = \frac{N - 100}{10} \]

So \( Z \) has approximately the standard normal distribution. Using this fact, we compute:

\[
P(100 - \Delta < N < 100 + \Delta) = P\left(100 - \Delta - 100 < N - 100 < 100 + \Delta - 100\right)
\]

\[
= P\left(-\frac{\Delta}{10} < \frac{N - 100}{10} < \frac{\Delta}{10}\right)
\]

\[
= P\left(-\frac{\Delta}{10} < Z < \frac{\Delta}{10}\right)
\]

\[
\approx \Phi\left(\frac{\Delta}{10}\right) - \Phi\left(-\frac{\Delta}{10}\right)
\]

\[
= \Phi\left(\frac{\Delta}{10}\right) - \left[1 - \Phi\left(\frac{\Delta}{10}\right)\right]
\]

The last equality follows from the symmetry about 0 of the standard normal distribution, i.e., \( \Phi(x) = 1 - \Phi(-x) \). Now we must solve for the value of \( \Delta \) such that the above probability is 0.90:

\[
\Phi\left(\frac{\Delta}{10}\right) - \left[1 - \Phi\left(\frac{\Delta}{10}\right)\right] = 0.9
\]

\[
\Rightarrow 2\Phi\left(\frac{\Delta}{10}\right) - 1 = 0.9
\]

\[
\Rightarrow 2\Phi\left(\frac{\Delta}{10}\right) = 1.9
\]

\[
\Rightarrow \Phi\left(\frac{\Delta}{10}\right) = 0.95
\]

\[
\Rightarrow \frac{\Delta}{10} = 1.645 \quad (\star)
\]

\[
\Rightarrow \Delta = 16.45
\]

Here the step indicated by (\(\star\)) is performed using a table of standard normal quantiles.