I. Basic Relations for Two Levels of Data (e.g., kids within classes)

For outcome $Y$, predictor $X$ and grouping variable $U$,

\[
\text{Var}(X) = \text{Var}(E(X|U)) + E(\text{Var}(X|U)),
\]

\[
\text{Cov}(Y, X) = \text{Cov}(E(X|U), E(Y|U)) + E(\text{Cov}(X, Y|U)).
\]

Define $\eta_X^2 = \frac{\text{Var}(E(X|U))}{\text{Var}(X)}$, $\beta_{YX}^t = \frac{\text{Cov}(Y, X)}{\text{Var}(X)}$,

\[
\beta_{YX}^b = \frac{\text{Cov}(E(X|U), E(Y|U))}{\text{Var}(E(X|U))}, \quad \beta_{YX}^{W-p} = \frac{E(\text{Cov}(Y, X|U))}{E(\text{Var}(X|U))}.
\]

Then

\[
\frac{\text{Cov}(Y, X)}{\text{Var}(X)} = \frac{\text{Cov}(E(X|U), E(Y|U))}{\text{Var}(X)} \left( \frac{\text{Var}(E(X|U))}{\text{Var}(E(X|U))} \right)
\]

\[
+ \frac{E(\text{Cov}(X, Y|U))}{\text{Var}(X)} \left( \frac{E(\text{Var}(X|U))}{E(\text{Var}(X|U))} \right).
\]

Regrouping and using the definitions above gives

\[
\beta_{YX}^t = \eta_X^2 \beta_{YX}^b + (1-\eta_X^2) \beta_{YX}^{W-p} \quad \text{(DCD)}.
\]
U, Z unobserved grouping vars, define indiv into group

\[ \beta_{YX}^t = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} ; \]  

\[ \beta_{YX}^b = \frac{\text{Cov}[E(X|U,Z), E(Y|U,Z)]}{\text{Var}(E(X|U,Z))} ; \]  

\[ \beta_{YX}^{w-p} = \frac{E(\text{Cov}(X, Y|U,Z))}{E(\text{Var}(X|U,Z))} . \]  

Another important quantity is the population intraclass correlation of X, commonly denoted by \( \eta_X^2 \), which is the ratio of the between-group variance and the total variance of X:

\[ \eta_X^2 = \frac{\text{Var}(E(X|U,Z))}{\text{Var}(X)} . \]  

\( \eta_X^2 \) is large (close to one) when the cases are quite homogeneous within each group but the group means are spread apart. When groups are formed by random assignment of cases to groups, \( \eta_X^2 = 0 \).

Starting with the decomposition of the total covariance, we form the equality:

\[ \text{Cov}(X, Y) = \text{Cov}[E(X|U,Z), E(Y|U,Z)] \left( \frac{\text{Var}(E(X|U,Z))}{\text{Var}(X)} \right) \left( \frac{\text{Var}(E(X|U,Z))}{\text{Var}(E(X|U,Z))} \right) \]

\[ + \frac{E(\text{Cov}(X, Y|U,Z))}{\text{Var}(X)} \left( \frac{E(\text{Var}(X|U,Z))}{E(\text{Var}(X|U,Z))} \right) . \]

Regrouping and using the definitions in equation (2) yields

\[ \beta_{YX}^t = \eta_X^2 \beta_{YX}^b + (1-\eta_X^2) \beta_{YX}^{w-p} . \]  

This familiar relation between the regression slopes was first presented by Duncan, Cuzzort, and Duncan (1961). Note that when \( \eta_X^2 \) is close to one, the between-group and total slopes are nearly equal in the population.