3.2 Fitting a Hierarchical Linear Model with \texttt{lm} \\

Following Bryk and Raudenbush (1992) and Singer (1998), I will fit a hierarchical linear model to the math-achievement data. This model consists of two equations: First, within schools, we have the regression of math achievement on the individual-level covariate SES; it aids interpretability of the regression coefficients to center SES at the school average; then the intercept for each school estimates the average level of math achievement in the school.

Using centered SES, the individual-level equation for individual \(j\) in school \(i\) is

\[
\text{mathach}_{ij} = \alpha_{0i} + \alpha_{1i} \text{mean SES} + \epsilon_{ij}
\]

At the school level, also following Bryk and Raudenbush, I will entertain the possibility that the school intercepts and slopes depend upon sector and upon the average level of SES in the schools:

\[
\begin{align*}
\alpha_{0i} &= \gamma_{00} + \gamma_{01} \text{mean SES}_i + \gamma_{02} \text{sector}_i + u_{0i} \\
\alpha_{1i} &= \gamma_{10} + \gamma_{11} \text{mean SES}_i + \gamma_{12} \text{sector}_i + u_{1i}
\end{align*}
\]

This kind of formulation is sometimes called a \textit{coefficients-as-outcomes} model.

Substituting the school-level equation 3 into the individual-level equation 2 produces

\[
\begin{align*}
\text{mathach}_{ij} &= \gamma_{00} + \gamma_{01} \text{mean SES}_i + \gamma_{02} \text{sector}_i + u_{0i} \\
&\quad + \left(\gamma_{10} + \gamma_{11} \text{mean SES}_i + \gamma_{12} \text{sector}_i + u_{1i}\right) \text{mean SES}_j + \epsilon_{ij}
\end{align*}
\]

Rearranging terms,

\[
\text{mathach}_{ij} = \gamma_{00} + \gamma_{01} \text{mean SES}_i + \gamma_{02} \text{sector}_i + \gamma_{10} \text{mean SES}_j + \gamma_{11} \text{mean SES}_i \times \text{mean SES}_j + \gamma_{12} \text{sector}_i \times \text{mean SES}_j + u_{0i} + u_{1i}\text{mean SES}_j + \epsilon_{ij}
\]

Here, the \(\gamma\)'s are fixed effects, while the \(u\)'s (and the individual-level errors \(\epsilon_{ij}\)) are random effects.

Finally, rewriting the model in the notation of the linear mixed model (equation 1),

\[
\text{mathach}_{ij} = \beta_1 + \beta_2 \text{mean SES}_i + \beta_3 \text{sector}_i + \beta_4 \text{mean SES}_j + \beta_5 \text{mean SES}_i \times \text{mean SES}_j + \beta_6 \text{sector}_i \times \text{mean SES}_j + b_{ij} + \epsilon_{ij}
\]

\(\text{lm} \) (linear mixed effects) function in the \texttt{nlme} library, however, employs the Laird-Ware form of the linear mixed model (after a seminal paper on the topic published by Laird and Ware, 1982):

\[
\begin{align*}
\psi_{ij} &= \beta_1 x_{1ij} + \cdots + \beta_p x_{pij} \\
&\quad + b_{1i} + \epsilon_{ij}
\end{align*}
\]

where

- \(y_{ij}\) is the value of the response variable for the \(j\)th of \(n_i\) observations in the \(i\)th of \(M\) groups or clusters.
- \(\beta_1, \ldots, \beta_p\) are the fixed-effect coefficients, which are identical for all groups.
- \(x_{1ij}, \ldots, x_{pij}\) are the fixed-effect regressors for observation \(j\) in group \(i\); the first regressor is usually for the constant, \(x_{1ij} = 1\).
- \(b_{1i}, \ldots, b_{p}\) are the random-effect coefficients for group \(i\), assumed to be multivariately normally distributed. The random effects, therefore, vary by group. The \(b_k\) are thought of as random variables, not as parameters, and are similar in this respect to the errors \(\epsilon_{ij}\).