Straight-line Growth Curve Formulation.

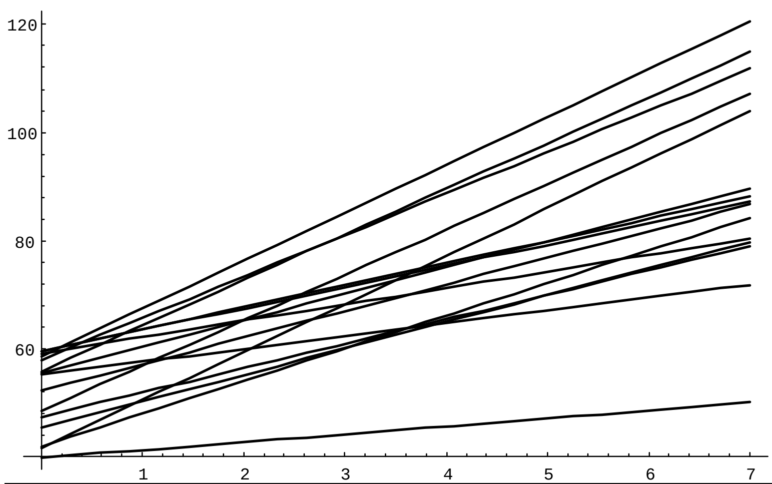
The within-unit model is a straight-line growth curve, the observables have the basic classical test theory measurement model, and the between-unit model has a single exogenous predictor (perfectly measured and most often with no missing data). Start with an attribute η which exhibits systematic change over time. For individual p in the population of individuals, denote the form of the growth curve in η for individual p as $\eta_p(t)$. A straight-line growth-curve is written as

$$\eta_p(t) = \eta_p(0) + \theta_p t \, . \qquad (1)$$

Rewrite (1) using the centering parameter t^o (from Rogosa and Willett, 1985, Sec. 2) which specifies a center for the time metric; $t^o = -\sigma_{\eta(0)\theta} / \sigma^2_\theta$. The centering parameter t^o has the convenient property that θ and $\eta(t^o)$ are uncorrelated over the population of individuals. Then in terms of the uncorrelated random variables $\eta(t^o)$ and θ the straight-line growth model is

$$\eta_p(t) = \eta_p(t^o) + \theta_p(t - t^o) \, . \qquad (1')$$

The constant rate of change θ_p is often the key parameter of interest in research questions about change. The parameters of the individual growth curves have a distribution over the population of individuals (often assumed to be Gaussian by default); the first two moments of the rate of change are written as μ_θ and σ^2_θ .



Shown at left 15 straight-line growth curves corresponding to pop. parameters $t^o = 2$; $\sigma^2_\theta = 5.333$; $\sigma^2_{\eta(t^o)} = 48$; $\theta \sim U[1, 9]$, $\eta(t^o) \sim U[38, 62]$. correlations among $\eta(t_i)$ for observation times $\rho_{\eta(1)\eta(4)} = .614$, $\rho_{\eta(1)\eta(6)} = .316$, $\rho_{\eta(4)\eta(6)} = .943$.

Longitudinal data sets including at least one exogenous (background) characteristic, denoted as W , allow pursuit of additional research questions about systematic individual differences in growth (i.e. correlates of change) and also to examine possible improvements in estimating growth curve parameters. The relation, over individuals, of W to the rate parameter θ, is summarized by the conditional expectation E(θ | W), stated here as the simplest possible straight-line regression

$$E(\theta \mid W) = \mu_\theta + \beta_{\theta W} (W - \mu_W) \quad , \qquad (2)$$

where the regression slope parameter is $\beta_{\theta W}$. Equation (2) is an example of a "between-unit" model. In the case where there is no measured exogenous variable, this between-unit model is $E(\theta \mid W) = \mu_\theta$. A similar relation can be stated for the intercept (aka "base" variable) in Equation 1 or 1' (e.g. predict $\eta_p(0)$, $\eta_p(t^o)$ or some particular $\eta_p(t)$ by the exogenous variable W).

Observables. Times of observation are $\{t_i\} = t_1 ,..., t_T$, which in these data analysis examples are the same for all p (except when observations at some t_i are missing for some p). From these discrete values of the times of observation, we then have values for the $\eta_p(t_i)$ for $p = 1, ..., n$. The completion of this set-up is the standard (oversimplified) statement that the observable Y is an imperfectly measured η, and the relation between Y and η is through the basic classical test theory model. $Y_p(t_i) = \eta_p(t_i) + \epsilon_i$ for $p = 1 , ..., n$. For convenience, the observables for individual p are written as $Y_{1p} ,..., Y_{Tp}$.
For $Y_p(t_i)$ generated from the collection of straight-line growth curves given above with var(ϵ) set to 5.0, the pop. correlations are $\rho_{Y(1)Y(4)}$ = .567, $\rho_{Y(1)Y(6)}$ = .297, $\rho_{Y(4)Y(6)}$ = .894.

It is convenient to consider W to be measured perfectly to conform to the common assumptions and especially to make $\beta_{\theta W}$ a main parameter of interest in the between-unit model (i.e., not distorted by measurement error in W).

Alternative: exponential growth to an asymptote

Exponential growth curve with asymptote λ_p and curvature δ

$$\eta_p(t) = \lambda_p - (\lambda_p - \eta_p(0))\exp(-\delta t) \quad .$$

Define a centering point t^o for exponential growth, where t^o has the properties that the variance of $\eta_p(t)$ is minimized at $t = t^o$ and that $\eta_p(t^o)$ and $(\lambda_p - \eta_p(t^o))$ are uncorrelated. For exponential growth

$t^o = (-1/\delta)\ln[(\sigma^2_\lambda - \sigma_{\lambda\,\eta(0)})/\sigma^2_{(\lambda - \eta(0))}$, and the growth curve can be written:

$$\eta_p(t) = \eta_p(t^o) + (\lambda_p - \eta_p(t^o))\,[1 - \exp(-\delta(t - t^o))] \quad .$$

Exponential Growth

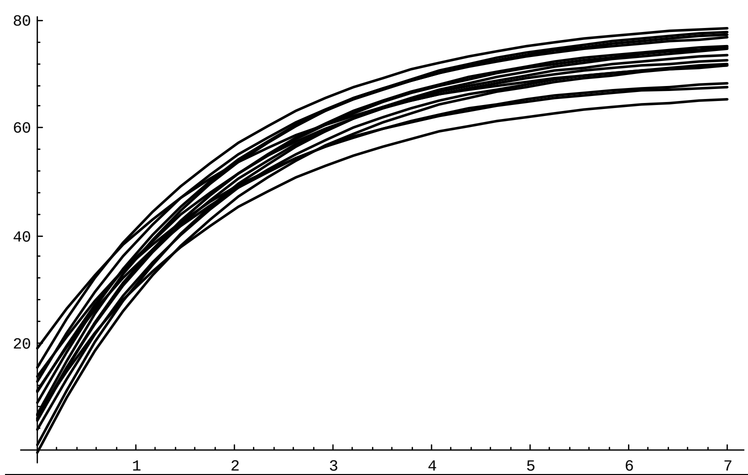


Figure above shows a collection of 15 exponential growth curves corresponding to population parameters

$t^o = 2$; $\delta = .5$; $\mu_\lambda = 75$; $\sigma^2_\lambda = 16$; $\mu_{\eta(t^o)} = 50$; $\sigma^2_{\eta(t^o)} = 9$ For reference, values of the population correlation coefficients among the $\eta(t_i)$ for observation times $t_1 = 1$, $t_2 = 3$, $t_3 = 5$ are: $\rho_{\eta(1)\eta(3)} = .657$, $\rho_{\eta(1)\eta(5)} = .435$, $\rho_{\eta(3)\eta(5)} = .965$.

exogenous variable W could be linked with both λ_p and $\eta_p(t^o)$.