

# Correct ordering in the Zipf–Poisson ensemble

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## Abstract

We consider a Zipf–Poisson ensemble in which  $X_i \sim \text{Poi}(Ni^{-\alpha})$  for  $\alpha > 1$  and  $N > 0$  and integers  $i \geq 1$ . As  $N \rightarrow \infty$  the first  $n'(N)$  random variables have their proper order  $X_1 > X_2 > \dots > X_{n'}$  relative to each other, with probability tending to 1 for  $n'$  up to  $(AN/\log(N))^{1/(\alpha+2)}$  for an explicit constant  $A(\alpha) \geq 3/4$ . The rate  $N^{1/(\alpha+2)}$  cannot be achieved. The ordering of the first  $n'(N)$  entities does not preclude  $X_m > X_{n'}$  for some interloping  $m > n'$ . The first  $n''$  random variables are correctly ordered exclusive of any interlopers, with probability tending to 1 if  $n'' \leq (BN/\log(N))^{1/(\alpha+2)}$  for  $B < A$ . For a Zipf–Poisson model of the British National Corpus, which has a total word count of 100,000,000, our result estimates that the 72 words with the highest counts are properly ordered.

## 1 Introduction

Power law distributions are ubiquitous, arising in studies of degree distributions of large networks, book and music sales counts, frequencies of words in literature and even baby names. It is common that the relative frequency of the  $i$ 'th most popular term falls off roughly as  $i^{-\alpha}$  for a constant  $\alpha$  slightly larger than 1. This is the pattern made famous by Zipf (1949). Data usually show some deviations from a pure Zipf model. The Zipf–Mandelbrot law for which the  $i$ 'th frequency is proportional to  $(i+k)^{-\alpha}$  where  $k \geq 0$  is often a much better fit. That and many other models are described in Popescu (2009, Chapter 9).

The usual methods for fitting long tailed distributions assume an IID sample. However, in many applications a persistent set of entities is repeatedly sampled under slightly different conditions. For example, if one gathers a large sample of English text, the word ‘the’ will be the most frequent word with overwhelming probability. No other word has such high probability and repeated sampling will not give either zero or two instances of such a very popular word. Similarly in Internet applications, the most popular URLs in one sample are likely to reappear in a similar sample, taken shortly thereafter or in a closely related stratum of users. The movies most likely to be rated at Netflix in one week will with only a few changes be the same popular movies the next week.

Because the entities themselves have a meaning beyond our sample data, it is natural to wonder whether they are in the correct order in our sample. The problem we address here is the error in ranks estimated from count data. By focussing on count data we are excluding other long tailed data such as the lengths of rivers.

In a large data set, the top few most popular items are likely to be correctly identified while the items that appeared only a handful of times cannot be confidently ordered from the sample. We are interested in drawing the line between ranks that are well identified and those that may be subject to sampling fluctuations. One of our motivating applications is a graphical display in Dyer (2010). Using that display one is able to depict a head to tail affinity for movie ratings data: the busiest raters are over represented in the most obscure movies and conversely the rare raters are over represented in ratings of very frequently rated movies. Both effects are concentrated in a small corner of the diagram. The graphic has greater importance if it applies to a source generating the data than if it applies only to the data at hand.

This paper uses the Zipf law because it is the simplest model for long tailed rank data and we can use it to get precisely stated asymptotic results. If instead we are given another distribution, then a numerical method described in Section 4 is very convenient to apply.

If we suppose that the item counts are independent and Poisson distributed with expectations that follow a power law, then a precise answer is possible. We define the Zipf–Poisson ensemble to be an infinite collection of independent random variables  $X_i \sim \text{Poi}(\lambda_i)$  where  $\lambda_i = Ni^{-\alpha}$  for parameters  $\alpha > 1$  and  $N > 0$ . Our main results are summarized in Theorem 1 below.

**Theorem 1.** *Let  $X_i$  be sampled from the Zipf–Poisson ensemble with parameter  $\alpha > 1$ . If  $n = n(N) \leq (AN/\log(N))^{1/(\alpha+2)}$  for  $A = \alpha^2(\alpha + 2)/4$ , then*

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_1 > X_2 > \dots > X_n) = 1. \quad (1)$$

*If  $n = n(N) \leq (BN/\log(N))^{1/(\alpha+2)}$  for  $B < A$ , then*

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_1 > X_2 > \dots > X_n > \max_{i>n} X_i) = 1. \quad (2)$$

*If  $n = n(N) \geq CN^{1/(\alpha+2)}$  for any  $C > 0$ , then*

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_1 > X_2 > \dots > X_n) = 0. \quad (3)$$

Equation (1) states that the top  $n' = \lfloor (AN/\log(N))^{1/(\alpha+2)} \rfloor$  entities, with  $A = \alpha^2(\alpha + 2)/4$ , are correctly ordered among themselves with probability tending to 1 as  $N \rightarrow \infty$ . From  $\alpha > 1$  we have  $A > 3/4$ . Equation (3) shows that we cannot remove  $\log(N)$  from the denominator, because the first  $CN^{1/(\alpha+2)}$  entities will fail to have the correct joint ordering with a probability approaching 1 as  $N \rightarrow \infty$ .

Equation (1) leaves open the possibility that some entity beyond the  $n'$ th manages to get among the top  $n'$  entities due to sampling fluctuations. Those

entities each have only a small chance to be bigger than  $X_{n'}$ , but there are infinitely many of them. Equation (2) shows that with probability tending to 1, the first  $n'' = \lfloor (BN/\log(N))^{1/(\alpha+2)} \rfloor$  entities are the correct first  $n''$  entities in the correct order. The limit holds for any  $B < A$ . That is, there is very little scope for interlopers.

Section 2 shows an example based on 100,000,000 words of the British National Corpus (BNC). See Aston and Burnard (1998). Using  $\alpha$  near 1.1 in the asymptotic formulas, we estimate that the first 72 words are correctly ordered among themselves. In a Monte Carlo simulation, very few interloping counts were seen. The estimate  $n' = 72$  depends on the Zipf-Poisson assumption which is an idealization, but it is quite stable if the log-log relationship is locally linear in a critical region of  $n$  values.

Section 3 proves our results. Of independent interest there is Lemma 1 which gives a Chernoff bound for the Skellam (1946) distribution: For  $\lambda \geq \nu > 0$  we show that  $\mathbb{P}(\text{Poi}(\lambda) \leq \text{Poi}(\nu)) \leq \exp(-(\sqrt{\lambda} - \sqrt{\nu})^2)$  where  $\text{Poi}(\lambda)$  and  $\text{Poi}(\nu)$  are independent Poisson random variables with the given means. Section 4 has our conclusions.

## 2 Example: the British National Corpus

Figure 1 plots the frequency of English words versus their rank on a log-log scale, for all words appearing at least 800 times among the approximately 100 million words of the BNC. The counts are from Kilgarrif (2006). The data plotted have a nearly linear trend with a slope just steeper than  $-1$ . They are not perfectly Zipf-like, but the fit is extraordinarily good considering that it uses just one parameter for 100 million total words.

The top 10 counts from Figure 1 are shown in Table 1. The most frequent word ‘the’ is much more frequent than the second most frequent word ‘be’. The process generating this data clearly favors the word ‘the’ over ‘be’ and a  $p$ -value for whether these words might be equally frequent, using Poisson assumptions is overwhelmingly significant. Though the 9’th and 10’th words have counts that are within a few percent of each other, they too are significantly different, as judged by  $(X_9 - X_{10})/\sqrt{X_9 + X_{10}} \doteq 34.9$ , the number of estimated standard deviations separating them. The 500’th and 501’st most popular words are ‘report’ and ‘pass’ with counts of 20,660 and 20,633 respectively. These are not significantly different.

We will use a value of  $\alpha$  close to 1.1 to illustrate the results of Theorem 1. The data appear to have approximately this slope in what will turn out to be the important region, with ranks from 10 to 100. We don’t know  $N$  but we can estimate it. Let  $T = \sum_{i=1}^{\infty} X_i$  be the total count. Then  $\mathbb{E}(T) = \sum_{i=1}^{\infty} Ni^{-\alpha} = N\zeta(\alpha)$  where  $\zeta(\cdot)$  is the Riemann zeta function. We find that  $\zeta(\alpha_*) = 10$  for  $\alpha_* \doteq 1.106$ . Choosing  $\alpha = \alpha_*$  we find that  $T = 10^8$  corresponds to  $N = N_* \equiv 10^7$ .

Theorem 1 has the top  $n' = (A(\alpha)N/(\log(N)))^{1/(\alpha+2)}$  entities correctly ordered among themselves with probability tending to 1. For the BNC data we

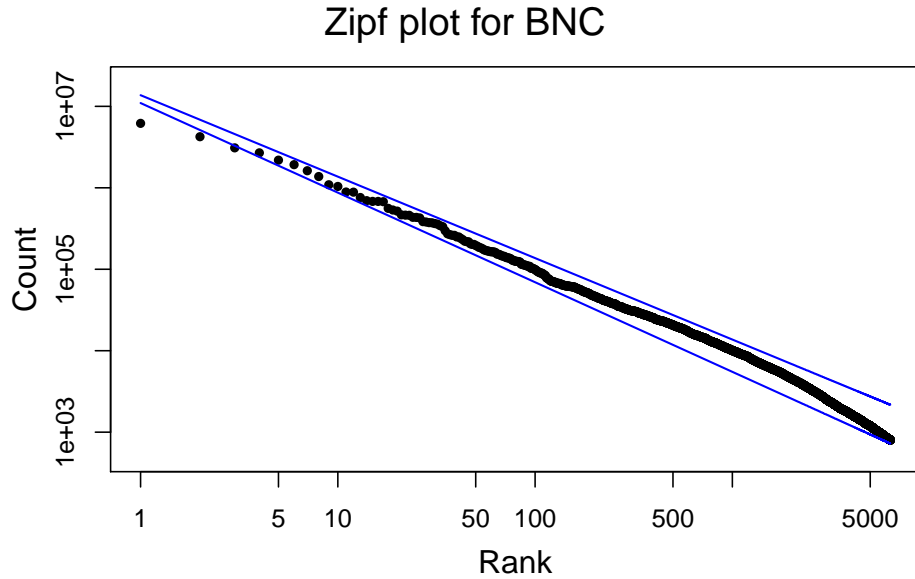


Figure 1: Zipf plot for the British National Corpus data. The reference line above the data has slope  $-1$ , while that below the data has slope  $-1.1$ .

Rank	Word	Count
1	the	6,187,267
2	be	4,239,632
3	of	3,093,444
4	and	2,687,863
5	a	2,186,369
6	in	1,924,315
7	to	1,620,850
8	have	1,375,636
9	it	1,090,186
10	to	1,039,323

Table 1: The top ten most frequent words from the British National Corpus, with their frequencies. Item 7 is the word ‘to’, used as an infinitive marker, while item 10 is ‘to’ used as a preposition. In “I went to the library to read.” the first ‘to’ is a preposition and the second is an infinitive marker.

## 1000 simulations of the Zipf–Poisson ensemble

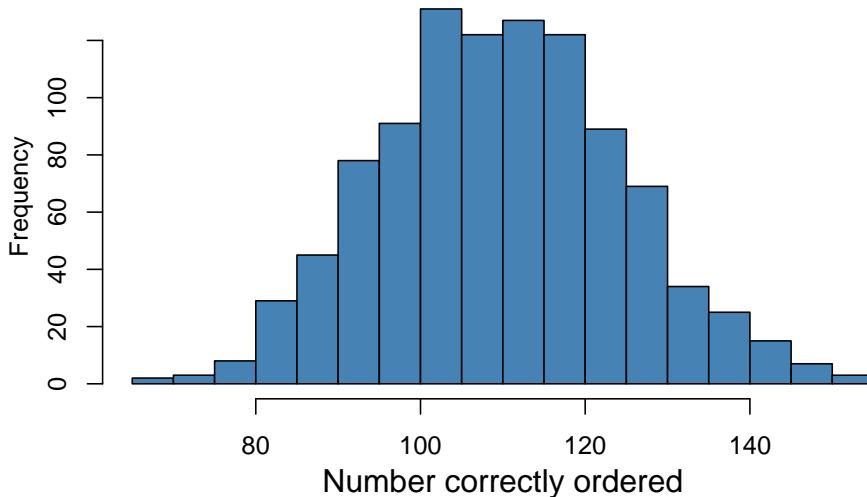


Figure 2: The Zipf–Poisson ensemble with  $N = 10^7$  and  $\alpha = 1.106$  was simulated 1000 times. The histogram shows the distribution of the number of correctly ordered words.

get  $n' = (A(\alpha_*)N_*/(\log(N_*)))^{1/(\alpha_*+2)} \doteq 72.08$ . For data like this, we could reasonably expect the true 72 most popular words to be correctly ordered among themselves.

We did a small simulation of the Zipf–Poisson model. The results are shown in Figure 2. The number of correctly ordered items ranged from 69 to 153 in those 1000 simulations. The number was only smaller than 72 for 2 of the simulated cases.

In our simulation, the first rank error to occur was usually a transposition between the  $n$ 'th and  $n + 1$ 'st entity. This happened 982 times. There were 7 cases with a tie between the  $n$ 'th and  $n + 1$ 'st entity. The remaining 11 cases all involved the  $n + 2$ 'nd entity getting ahead of the  $n$ 'th. As a result, we see that interlopers are very rare, as we might expect from Lemma 4.

Lack of fit of the Zipf–Poisson model will affect the estimate somewhat. Here we give a simple analysis to show that the estimate  $n' = 72$  is remarkably stable. Even though the log–log plot in Figure 1 starts off shallower than slope  $-1$ , the first 10 counts are so large that we can be confident that they are correctly ordered. Similarly, the log–log plot ends up somewhat steeper than  $-1.1$ , but that takes place for very rare items that have negligible chance of ending up ahead of the 72'nd word. As a result we choose to work with  $\alpha = 1.106$  and re-estimate  $N$  to match the counts in the range  $10 \leq n \leq 100$ . Those counts are large and very stable. A simple estimate of  $N$  is  $N_i = X_i i^\alpha$ . For this data  $\min_{10 \leq i \leq 100} N_i \doteq 1.25 \times 10^7$ . Using  $N = 1.25 \times 10^7$  with  $\alpha = 1.106$  and  $B = 1$

### Test statistics for order

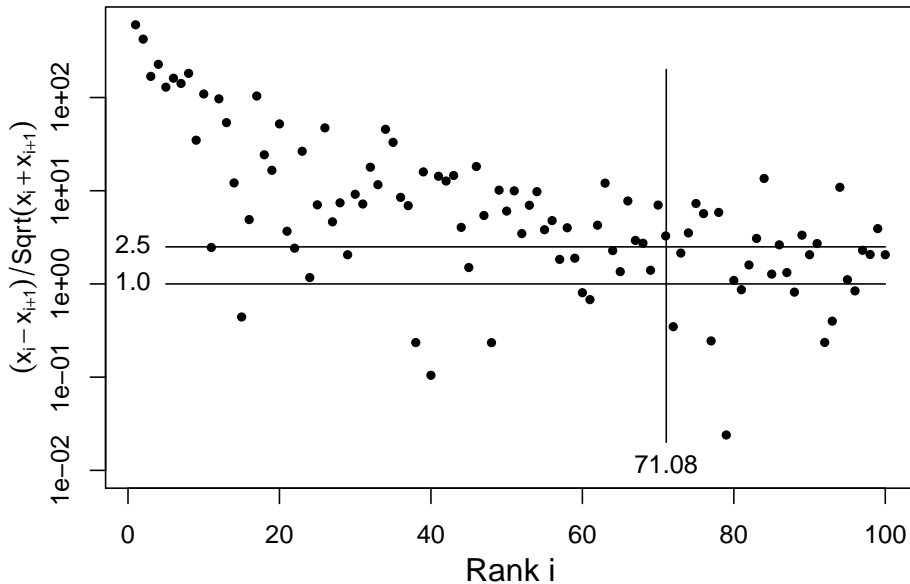


Figure 3: This figure plots  $(X_i - X_{i+1})/\sqrt{X_i + X_{i+1}}$  versus  $i = 1, \dots, 100$  for the BNC data. The horizontal reference lines are drawn at 2.5 standard errors and at 1.0 standard errors. The vertical line is drawn at  $n' - 1 = 71.08$ .

gives  $n' \doteq 77.10$  raising the estimate only slightly from 72. The value  $\alpha_* = 1.106$  was chosen partly based on fit and partly based on numerical convenience that  $\zeta(\alpha_*) = 10$ . Repeating our computation with  $1.05 \leq \alpha \leq 1.15$  gives values of  $N$  that range from  $1.1 \times 10^7$  to  $1.4 \times 10^7$ , and estimates  $n'$  from 71.04 to 77.29. This estimate is very stable because the Zipf curve is relatively straight in the critical region.

There is enough wiggling in the log-log plot Figure 1 between ranks 10 and 50, that can be attributed to  $\mathbb{E}(X_i)$  not perfectly following a local power law there. The British National Corpus rank orderings are not quite as reliable as those in the fitted Zipf-Poisson model. Unsurprisingly, a one parameter model shows some lack of fit on this enormous data set.

Figure 3 plots standard errors for the first 100 consecutive word comparisons. A horizontal line is at 2.5. The theorem predicts that the first  $n' = 72.08 \doteq 72$  words would be correctly ordered relative to each other. When the first  $n'$  words are correctly ordered the first  $n' - 1$  differences have the correct sign. The vertical reference line is at 71.08. Beyond 72 it is clear that many consecutive word orderings are doubtful. We also see a few small standard errors for  $n < 72$  which correspond to some local flat spots in Figure 1. As a result we might expect a small number of transposition errors among the first 72 words, in addition to the large number of rank errors that set in beyond the 72'nd word,

as predicted by the Zipf–Poisson model.

### 3 Proof of Theorem 1

Theorem 1 has three claims. First, equation (1) on correct ordering of the  $n$  most popular items within themselves, follows from Corollary 1 below. Combining that corollary with Lemma 4 to rule out interlopers, establishes (2) in which the first  $n$  items are correctly identified and ordered. The third claim (3), showing the necessity of the logarithmic factor, follows from Corollary 2.

#### 3.1 Some useful inequalities

The proof of Theorem 1 makes use of some bounds on Poisson probabilities and the gamma function, collected here.

Let  $Y \sim \text{Poi}(\lambda)$ . Shorack and Wellner (1986, page 485) have the following exponential bounds

$$\mathbb{P}(Y \geq t) \leq \left(1 - \frac{\lambda}{t+1}\right)^{-1} \frac{e^{-\lambda} \lambda^t}{t!} \quad \text{for integers } t \geq \lambda, \quad \text{and} \quad (4)$$

$$\mathbb{P}(Y \leq t) \leq \left(1 - \frac{t}{\lambda}\right)^{-1} \frac{e^{-\lambda} \lambda^t}{t!} \quad \text{for integers } t < \lambda. \quad (5)$$

Klar (2000) shows that (4) holds for  $t \geq \lambda - 1$ . Equation (4) holds for real valued  $t \geq \lambda$  and equation (5) also holds for real valued  $t < \lambda$ . In both cases we interpret  $t!$  as  $\Gamma(t+1)$ .

A classic result of Teicher (1955) is that

$$\mathbb{P}(Y \leq \lambda) \geq \exp(-1) \quad (6)$$

when  $Y \sim \text{Poi}(\lambda)$ . If  $Y \sim \text{Poi}(\lambda)$ , then

$$\sup_{-\infty < t < \infty} \left| \mathbb{P}\left(\frac{Y - \lambda}{\sqrt{\lambda}} \leq t\right) - \Phi(t) \right| \leq \frac{0.8}{\sqrt{\lambda}}, \quad (7)$$

where  $\Phi$  is the standard normal CDF. Equation (7) follows by specializing a Berry-Esseen result for compound Poisson distributions (Michel, 1993, Theorem 1) to the case of a Poisson distribution.

We will also use Gautschi’s (1959) inequality on the Gamma function,

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s} \quad (8)$$

which holds for  $x > 0$  and  $0 < s < 1$ .

#### 3.2 Correct relative ordering, equation (1)

The difference of two independent Poisson random variables has a Skellam (1946) distribution. We begin with a Chernoff bound for the Skellam distribution.

**Lemma 1.** Let  $Z = X - Y$  where  $X \sim \text{Poi}(\lambda)$  and  $Y \sim \text{Poi}(\nu)$  are independent and  $\lambda \geq \nu$ . Then

$$\mathbb{P}(Z \leq 0) \leq \exp(-(\sqrt{\lambda} - \sqrt{\nu})^2). \quad (9)$$

*Proof.* Let  $\varphi(t) = \lambda e^{-t} + \nu e^t$ . Then  $\varphi$  is a convex function attaining its minimum at  $t^* = \log(\sqrt{\lambda/\nu}) \geq 0$ , with  $\varphi(t^*) = 2\sqrt{\lambda\nu}$ . Using the Laplace transform of the Poisson distribution

$$m(t) \equiv \mathbb{E}(e^{-tZ}) = e^{\lambda(e^{-t}-1)} e^{\nu(e^t-1)} = e^{-(\lambda+\nu)\varphi(t)}.$$

For  $t \geq 0$ , Markov's inequality gives  $\mathbb{P}(Z \leq 0) = \mathbb{P}(e^{-tZ} \geq 1) \leq \mathbb{E}(e^{-tZ})$ . Taking  $t = t^*$  yields (9).  $\square$

**Lemma 2.** Let  $X_i$  be sampled from the Zipf-Poisson ensemble. Then for  $n \geq 2$ ,

$$\mathbb{P}(X_1 > X_2 > \cdots > X_n) \geq 1 - n \exp\left(-\frac{N\alpha^2}{4} n^{-\alpha-2}\right). \quad (10)$$

*Proof.* By Lemma 1 and the Bonferroni inequality, the probability that  $X_{i+1} \geq X_i$  holds for any  $i < n$  is no more than

$$\sum_{i=1}^{n-1} \exp(-(\sqrt{\lambda_i} - \sqrt{\lambda_{i+1}})^2) = \sum_{i=1}^{n-1} \exp(-N(\sqrt{\theta_i} - \sqrt{\theta_{i+1}})^2). \quad (11)$$

For  $x \geq 1$ , let  $f(x) = x^{-\alpha/2}$ . Then  $|\sqrt{\theta_i} - \sqrt{\theta_{i+1}}| = |f(i) - f(i+1)| = |f'(z)|$  for some  $z \in (i, i+1)$ . Because  $|f'|$  is decreasing, (11) is at most  $n \exp(-Nf'(n)^2)$ , establishing (10).  $\square$

Now we can establish the first claim in Theorem 1.

**Corollary 1.** Let  $X_i$  be sampled from the Zipf-Poisson ensemble. Choose  $n = n(N) \geq 2$  so that  $n \leq (AN/\log(N))^{1/(\alpha+2)}$  holds for all large enough  $N$  where  $A = \alpha^2(\alpha+2)/4$ . Then

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_1 > X_2 > \cdots > X_n) = 1.$$

*Proof.* For large enough  $N$  we let  $n = (A_N N/\log(N))^{1/(\alpha+2)}$  for  $A_N \leq A$ . Then

$$\begin{aligned} n \exp\left(-\frac{N\alpha^2}{4} n^{-\alpha-2}\right) &= \left(\frac{A_N N}{\log(N)}\right)^{1/(\alpha+2)} N^{-\alpha^2/(4A_N)} \\ &= \left(\frac{A_N N}{\log(N)}\right)^{1/(\alpha+2)} N^{-\alpha^2/(4\alpha^2(\alpha+2)/4)} \\ &\leq \left(\frac{A}{\log(N)}\right)^{1/(\alpha+2)} \\ &\rightarrow 0. \end{aligned}$$

The proof then follows from Lemma 2.  $\square$



### 3.3 Correct absolute ordering, equation (2)

For the second claim in Theorem 1 we need to control the probability that one of the entities  $X_i$  from the tail given by  $i > n$ , can jump over one of the first  $n$  entities. Lemma 3 bounds the probability that an entity from the tail of the Zipf–Poisson ensemble can jump over a high level  $\tau$ .

**Lemma 3.** *Let  $X_i$  for  $i \geq 1$  be from the Zipf–Poisson ensemble with parameter  $\alpha > 1$ . If  $\tau \geq \lambda_n$  then*

$$\mathbb{P}\left(\max_{i>n} X_i > \tau\right) \leq \frac{N^{1/\alpha}}{\alpha} \frac{\tau + 1}{\tau + 1 - \lambda_n} \frac{\tau^{-1/\alpha}}{\tau - 1/\alpha}. \quad (12)$$

*Proof.* First,  $\mathbb{P}(\max_{i>n} X_i > \tau) \leq \sum_{i=n+1}^{\infty} \mathbb{P}(X_i > \tau)$  and then from (4)

$$\mathbb{P}\left(\max_{i>n} X_i > \tau\right) \leq \left(1 - \frac{\lambda_n}{\tau + 1}\right)^{-1} \sum_{i=n+1}^{\infty} \frac{e^{-\lambda_i} \lambda_i^\tau}{\Gamma(\tau + 1)}.$$

Now  $\lambda_i = Ni^{-\alpha}$ . For  $i > n$  we have  $\tau > \lambda_i = Ni^{-\alpha}$ . Over this range,  $e^{-\lambda} \lambda^\tau$  is an increasing function of  $\lambda$ . Therefore,

$$\begin{aligned} \sum_{i=n+1}^{\infty} e^{-\lambda_i} \lambda_i^\tau &\leq \int_n^{\infty} e^{-Nx^{-\alpha}} (Nx^{-\alpha})^\tau dx \\ &\leq \frac{N^{1/\alpha}}{\alpha} \int_0^{Nn^{-\alpha}} e^{-y} y^{\tau-1/\alpha-1} dy \\ &\leq \frac{N^{1/\alpha}}{\alpha} \Gamma(\tau - 1/\alpha). \end{aligned}$$

As a result

$$\mathbb{P}\left(\max_{i>n} X_i > \tau\right) \leq \frac{N^{1/\alpha}}{\alpha} \frac{\tau + 1}{\tau + 1 - \lambda_n} \frac{\Gamma(\tau - 1/\alpha)}{\Gamma(\tau + 1)}.$$

Now

$$\frac{\Gamma(\tau - 1/\alpha)}{\Gamma(\tau + 1)} = \frac{\Gamma(\tau + 1 - 1/\alpha)}{\Gamma(\tau + 1)} \frac{1}{\tau - 1/\alpha} < \frac{\tau^{-1/\alpha}}{\tau - 1/\alpha}$$

by Gautschi's inequality (8), with  $s = 1 - 1/\alpha$ , establishing (12).  $\square$

For an incorrect ordering to arise, either an entity from the tail exceeds a high level, or an entity from among the first  $n$  is unusually low. Lemma 4 uses a threshold for which both such events are unlikely, establishing the second claim (2) of Theorem 1.

**Lemma 4.** *Let  $X_i$  for  $i \geq 1$  be from the Zipf–Poisson ensemble with parameter  $\alpha > 1$ . Let  $n(N)$  satisfy  $n \geq (AN/\log(N))^{1/(\alpha+2)}$  for  $0 < A < A(\alpha) = \alpha^2(\alpha + 2)/4$ . Let  $m \leq (BN/\log(N))^{1/(\alpha+2)}$  for  $0 < B < A$ . Then*

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\max_{i>n} X_i \geq X_m\right) = 0. \quad (13)$$

*Proof.* For any threshold  $\tau$ ,

$$\mathbb{P}\left(\max_{i>n} X_i \geq X_m\right) \leq \mathbb{P}\left(\max_{i>n} X_i > \tau\right) + \mathbb{P}(X_m \leq \tau). \quad (14)$$

The threshold we choose is  $\tau = \sqrt{\lambda_m \lambda_n}$  where  $\lambda_i = \mathbb{E}(X_i) = Ni^{-\alpha}$ .

Write  $n = (A_N N / \log(N))^{1/(\alpha+2)}$  and  $m = (B_N N / \log(N))^{1/(\alpha+2)}$  for  $0 < B_N < B < A_N < A < A(\alpha)$ . Then  $\tau = \sqrt{\lambda_m \lambda_n} = N(C_N N / \log(N))^{-\alpha/(\alpha+2)}$  where  $C_N = \sqrt{A_N B_N}$ . Therefore

$$\tau = O(N^{2/(\alpha+2)}(\log(N))^{\alpha/(\alpha+2)}).$$

By construction,  $\tau > \lambda_n$  and so by Lemma 3

$$\mathbb{P}\left(\max_{i>n} X_i > \tau\right) \leq \frac{N^{1/\alpha}}{\alpha} \frac{\tau + 1}{\tau + 1 - \lambda_n} \frac{\tau^{-1/\alpha}}{\tau - 1/\alpha}.$$

Because  $\lambda_n/\tau = (B_N/A_N)^{\alpha/(2\alpha+4)}$ , we have  $(\tau + 1)/(\tau + 1 - \lambda_n) = O(1)$ . Therefore

$$\mathbb{P}\left(\max_{i>n} X_i > \tau\right) = O(N^{1/\alpha} \tau^{-1/\alpha-1}) = O(N^{-1/(\alpha+2)}(\log(N))^{(\alpha+1)/(\alpha+2)})$$

and so the first term in (14) tends to 0 as  $N \rightarrow \infty$ .

For the second term in (14), notice that  $X_m$  has mean  $\lambda_m > \tau$  and standard deviation  $\sqrt{\lambda_m}$ . Letting  $\rho = \alpha/(\alpha+2)$  and applying Chebychev's inequality, we find that

$$\begin{aligned} \mathbb{P}(X_m \leq \tau) &\leq \frac{\lambda_m}{(\tau - \lambda_m)^2} \\ &= \frac{B_N^\rho}{(B_N^\rho - C_N^\rho)^2} N^{-2/(\alpha+2)} (\log(N))^{-\rho} \\ &\leq \frac{1}{(A^{\rho/2} - B^{\rho/2})^2} N^{-2/(\alpha+2)} (\log(N))^{-\rho} \\ &\rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ . □

Lemma 4 is sharp enough for our purposes. A somewhat longer argument in Dyer (2010) shows that the interloper phenomenon is ruled out even deeper into the tail of the Zipf-Poisson ensemble. Specifically, if  $m \leq (BN)^\beta$  and  $n \geq (AN)^\beta$  for  $0 < B < A$  and  $\beta < 1/\alpha$ , then (13) still holds.

### 3.4 Limit to correct ordering, equation (3)

While we can get  $(AN/\log(N))^{1/(\alpha+2)}$  entities properly ordered, there is a limit to the number of correctly ordered entities. We cannot get above  $CN^{1/(\alpha+2)}$  correctly ordered entities, asymptotically. That is, the logarithm cannot be removed. We begin with a lower bound on the probability of a wrong ordering for two consecutive entities.

**Lemma 5.** *Let  $X_i$  be from the Zipf–Poisson ensemble with  $\alpha > 1$ . Suppose that  $AN^{1/(\alpha+2)} \leq i < i+1 \leq BN^{1/(\alpha+2)}$  where  $0 < A < B < \infty$ . Then for large enough  $N$ ,*

$$\mathbb{P}(X_{i+1} \geq X_i) \geq \frac{1}{3} \Phi\left(-\alpha \frac{A^{\alpha/2}}{B^{\alpha+1}}\right).$$

*Proof.* First  $\mathbb{P}(X_{i+1} \geq X_i) \geq \mathbb{P}(X_{i+1} > \lambda_i)\mathbb{P}(X_i \leq \lambda_i) \geq \mathbb{P}(X_{i+1} > \lambda_i)/e$  using Teicher’s inequality (6). Next

$$\mathbb{P}(X_{i+1} > \lambda_i) = 1 - \mathbb{P}(X_{i+1} \leq \lambda_i) \geq \Phi\left(\frac{\lambda_{i+1} - \lambda_i}{\sqrt{\lambda_{i+1}}}\right) - \frac{0.8}{\sqrt{\lambda_{i+1}}}.$$

Now,

$$\frac{\lambda_{i+1} - \lambda_i}{\sqrt{\lambda_{i+1}}} = \sqrt{N} \frac{(i+1)^{-\alpha} - i^{-\alpha}}{\sqrt{(i+1)^{-\alpha}}} = -\alpha\sqrt{N} \frac{(i+\eta)^{-\alpha-1}}{\sqrt{(i+1)^{-\alpha}}}$$

for some  $\eta \in (0, 1)$ . Applying the bounds on  $i$ ,

$$\frac{\lambda_{i+1} - \lambda_i}{\sqrt{\lambda_{i+1}}} \geq -\alpha\sqrt{N} \frac{(N^{1/(\alpha+2)}A)^{\alpha/2}}{(N^{1/(\alpha+2)}B)^{\alpha+1}} = -\alpha \frac{A^{\alpha/2}}{B^{\alpha+1}}.$$

Finally, letting  $N \rightarrow \infty$  we have  $\lambda_{i+1} \rightarrow \infty$  and so  $0.8/\sqrt{\lambda_{i+1}}$  is eventually smaller than  $(1 - e/3)\Phi(-\alpha A^{\alpha/2}B^{-\alpha-1})$ . Letting  $\theta = -\alpha A^{\alpha/2}B^{-\alpha-1}$  we have, for large enough  $N$ ,

$$\mathbb{P}(X_{i+1} \geq X_i) \geq \left(\Phi(\theta) - \left(1 - \frac{e}{3}\right)\Phi(\theta)\right)\frac{1}{e} = \frac{1}{3}\Phi(\theta). \quad \square$$

To complete the proof of Theorem 1 we establish equation (3). For  $n$  beyond a multiple of  $N^{1/(\alpha+2)}$ , the reverse orderings predicted by Lemma 5 cannot be avoided.

**Corollary 2.** *Let  $X_i$  be sampled from the Zipf–Poisson ensemble. Suppose that  $n = n(N)$  satisfies  $n \geq CN^{1/(\alpha+2)}$  for  $0 < C < \infty$ . Then*

$$\lim_{N \rightarrow \infty} \mathbb{P}(X_1 > X_2 > \cdots > X_n) = 0.$$

*Proof.* Let  $p \in (0, 1)$  be a constant such that  $\mathbb{P}(X_{i+1} \geq X_i) \geq p$  holds for all large enough  $N$  and  $(C/2)N^{1/(2+\alpha)} \leq i < i+1 \leq CN^{1/(2+\alpha)}$ . For instance Lemma 5 shows that  $p = \Phi(-\alpha(C/2)^{\alpha/2}/C^\alpha)/3 = \Phi(-\alpha(2C)^{-\alpha/2})/3$  is such a constant. Then

$$\mathbb{P}(X_1 > X_2 > \cdots > X_n) \leq \prod_i^* \mathbb{P}(X_i > X_{i+1}) \quad (15)$$

holds where  $\prod^*$  is over all odd integers  $i \in [(C/2)N^{1/(\alpha+2)}, CN^{1/(\alpha+2)}]$ . There are roughly  $CN^{1/(\alpha+2)}/4$  odd integers in the product. For large enough  $N$ , the right side of (15) is below  $(1 - p)^{CN^{1/(\alpha+2)}/5} \rightarrow 0$ .  $\square$

## 4 Discussion

We have found that the top few entities in a Zipf plot of counts can be expected to be in the correct order, when their frequencies are measured with Poisson errors. Even in the idealized Zipf setting, the number of correctly ordered entities grows fairly slowly with  $N$ .

Our transition point is at  $n' = (\alpha^2(\alpha+2)N/(4 \log(N)))^{1/(\alpha+2)}$  and estimating  $N$  from  $T = \sum_i X_i$  leads to the estimate

$$\hat{n} = \left( \frac{\alpha^2(\alpha+2)T/\zeta(\alpha)}{4 \log(T/\zeta(\alpha))} \right)^{\frac{1}{\alpha+2}}.$$

The threshold  $n'$  uses some slightly conservative estimates to get a rate in  $N$ . For the Zipf–Poisson ensemble with  $N = 10^7$  and  $\alpha = 1.106$  we can use (10) of Lemma 2 directly to find

$$1 - \mathbb{P}(X_1 > X_2 > \dots > X_{72}) \leq \sum_{i=1}^{71} \exp(-N(i^{-\alpha/2} - (i+1)^{-\alpha/2})^2) \doteq 0.0199.$$

We get a bound of 1% by taking  $n = 70$  and a bound of 5% by taking  $n = 76$ . The formula for  $\hat{n}$  comes remarkably close to what we get working directly with equation (10).

The Skellam bounds do not assume a Zipf rate for the Poisson means. Therefore we can use them to generalize the computation above. For example, with a Zipf–Mandelbrot–Poisson ensemble having  $X_i \sim \text{Poi}(N(i+k)^{-\alpha})$  we can still apply equation (10) to show that the probability of an error among the first  $n$  ranks is at most

$$p(n; N, \alpha, k) = \sum_{i=1}^{n-1} \exp\left(-N\left((i+k)^{-\alpha/2} - (i+k+1)^{-\alpha/2}\right)^2\right). \quad (16)$$

A conservative estimate of the number of correct positions in the Zipf–Mandelbrot–Poisson ensemble is

$$n' = \max\{n \geq 1 \mid p(n; N, \alpha, k) \leq 0.01\} \quad (17)$$

with  $n' = 0$  if  $p(1; N, \alpha, k) > 0.01$ . We can estimate  $N$  by  $T/\zeta(\alpha, k-1)$  where  $T = \sum_i X_i$  and  $\zeta(\alpha, h) = \sum_{\ell=0}^{\infty} (\ell+h)^{-\alpha}$  is the Hurwitz zeta function.

Equation (17) is conservative because it stems from the Bonferroni inequality, and does not adjust for two or more order relations being violated. It will be less conservative for small target probabilities like 0.01 than for large ones where adjustments are relatively more important.

Our focus is on the ranks that are correctly estimated. Methods to estimate parameters of the Zipf distribution or Zipf–Mandelbrot distribution typically use values of  $X_i$  for  $i$  much larger than the number of correctly identified items. It is not unreasonable to do so, because ordering errors tend to distribute the values of  $X_i$  both above and below the parametric trend line.

A small number of correct unique words can correspond to a reasonably large fraction of word usage. The BNC is roughly 6.2% ‘the’ and the top 72 words comprise about 45.3% of the corpus.

For large  $N$ , the top  $n_\epsilon = N^{1/(\alpha+2)-\epsilon}$  entities get properly ordered with very high probability for  $0 < \epsilon < 1/(\alpha + 2)$ . The tail beyond  $n_\epsilon$  accounts for a proportion of data close to  $\zeta(\alpha)^{-1} \int_{n_\epsilon}^{\infty} x^{-\alpha} dx = O(n_\epsilon^{-\alpha+1}) = O(N^{(1-\alpha)/(\alpha+2)+\epsilon'})$  for  $\epsilon' = \epsilon(\alpha - 1)$ . Taking small  $\epsilon$  and recalling that  $\alpha > 1$  we find that the fraction of data from improperly ordered entities vanishes in the Zipf–Poisson ensemble. When  $\alpha$  is just barely larger than 1 the rate may be slow.

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