Confidence intervals with control of the sign error in low power settings

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Abstract
When hypothesis tests of $H_0 : \theta = 0$ have low power, it is possible that their rejection can frequently be accompanied by an estimate $\hat{\theta}$ that has the wrong sign and significantly exaggerated magnitude. Such sign errors are less likely when the confidence interval for $\theta$ is well separated from 0, as measured in units of the confidence interval's width. Sign errors can be controlled by declaring confidence in the sign of $\theta$ only when $H_0$ is also rejected at a smaller level $\alpha_s = 2\alpha_1\alpha_S$, where $\alpha_S \leq 1/2$ is a user specified upper bound on the probability of a sign error given that $H_0$ has been rejected. This procedure is a very simple form of inference after model selection: the selected model is that $\theta \neq 0$ and the second inference is then on sign($\theta$).

1 Introduction
A serious problem arises for hypothesis tests of low statistical power, as pointed out by Gelman (2014). Low power arises when sampling uncertainty is large compared to the effect size. Statistical significance requires the estimated effect size to be a reasonable multiple of the sampling uncertainty. Therefore all of the observed significant outcomes exaggerate the true effect size, in this low power setting.

The issue typically arises in the context of testing a null hypothesis $H_0 : \theta = 0$ for some parameter $\theta$ of interest. It is usually implausible that $\theta$ could be exactly zero in a real world context. A common justification for testing $H_0$ anyway is that when you reject it, at least you have good information about the sign of $\theta$. If $\theta > 0$, then testing against $\theta = -\epsilon$ is a relatively hard negative value to distinguish from $\theta$ and $H_0$ emerges in the limit as $\epsilon \to 0$.

Gelman’s example shows that in the low power case, rejecting the point null $H_0$ can not only exaggerate the magnitude of $\theta$, it can also leave one with an incorrect sign. The point was made earlier by Gelman and Tuerlinckx (2000) who showed that some hierarchical Bayesian credible intervals greatly diminish the sign error problem, although their inferences are more conservative.
One way to handle the problem is to avoid doing tests that have low power. We should instead use a large sample or some other informative design and only study effects that are large. However we often cannot be sure what the power is going to be before doing a study and so low power settings will inevitably arise. Another approach is to employ Bayes factors, as for example Bayarri et al. (2016) do in a Bayes-frequentist hybrid. That will not suit every user as specifying priors can be difficult.

Confidence intervals have the benefit of being in the same units as the effect $\theta$ and hence they facilitate study of practical significance. The approach proposed here is to only draw conclusions about the sign of $\hat{\theta}$ when a confidence interval for $\theta$ is not very wide compared to the distance from $\hat{\theta}$ to zero. This method takes explicit account of the apparent magnitude of $\theta$, and not just whether the interval includes zero. In low power settings, many rejections of $H_0$ will be unaccompanied by a decision about the sign of $\theta$. In high throughput testing of many hypotheses, some but not all rejections may be accompanied by decisions about the sign.

The approach is an extremely basic form of conditional inference as described by Fithian et al. (2014) who also give a comprehensive bibliography. We reject $H_0$ at some level $\alpha_1$ to conclude that $\theta \neq 0$ and if we can also reject $H_0$ at a more stringent level $\alpha_2$, then we additionally conclude that $\text{sign}(\theta) = \text{sign}(\hat{\theta})$.

An outline of this paper is as follows. Section 2 gives the problem formulation and shows how to compute the standard error of $\hat{\theta}$ given the level $\alpha$ and power $1 - \beta$ of a test of $H_0 : \theta = 0$ when the true $\theta$ is one unit. It seems that there is an error in the figure from Gelman (2014) but that his substantive points remain valid. After an email conversation with Andrew Gelman, it appears that figure illustrated a data set in which the power was just over 5.5% which then rounds up to 6%. Section 3 shows how sign errors and exaggeration factors vary with the underlying power when $\alpha = 0.05$. That is not to say 0.05 is a good choice. It is however, the popular one. Sign errors become very unlikely when the power is over 30% but magnitude errors can be substantial at higher powers. Section 4 proposes a way to control sign errors conditionally on rejecting $H_0$. We motivate it by comparing the confidence interval width to the distance of $\hat{\theta}$ from 0. The idea is to choose two significance levels $\alpha_1$ and $\alpha_S < 1/2$. If $H_0$ can be rejected at level $\alpha_1$ conclude that $\theta \neq 0$. If $H_0$ can also be rejected at the level $\alpha_2 = 2\alpha_1\alpha_S$, conclude that we also know the sign of $\theta$ equals that of $\hat{\theta}$. The probability of a sign error given that $H_0$ has been rejected is then at most $\alpha_S$. The power for conclusions that come with a decision on the sign of $\theta$ is necessarily lower than the power for simply rejecting $H_0$. This seems appropriate. It is also possible to define a $p$-value for the sign as the smallest $\alpha_S$ for which $H_0$ is rejected at level $\alpha_2$, although this approach requires that $\alpha_1$ be prespecified and not a researcher degree of freedom. Section 5 looks at one-tailed tests. They do not solve the problem. For the same effect size that has 6 percent power, running a one-tailed test at level $\alpha = 0.05$ would (falsely) reject $H_0$ just over 26% of the time.
$$\tau = \text{function}(\alpha, \text{powr})\{$$

# A Gaussian test at level alpha
# has power powr for Y ~ N( 1, \tau^2 ).
# Solve for tau using noncentral chisquare.

$$\text{aux} = \text{function}(\tau)\{$$
$$1 - \text{powr} - \text{pchisq(qchisq(1-\alpha,1),1,ncp=1/\tau^2)}$$
$$\}$$

$$\text{ur} = \text{uniroot( aux, lower=10^{-9},upper=10^{6},}$$
$$\text{tol = .Machine$double.eps^0.9)}$$

$$\text{ur$root}$$

Figure 1: R code to compute the critical standard error $\tau$.

2 Problem formulation

We focus on the large sample setting where the statistical estimate is approximately normally distributed. Suppose that $\theta \in \mathbb{R}$ is an effect that we estimate by a statistic $\hat{\theta}$. There is a null hypothesis $H_0$ that $\theta = \theta_0$. The most common case has $\theta_0 = 0$ which we assume from here on. We reject $H_0$ for a large $|\hat{\theta}|$.

Let $\hat{\theta} \sim \mathcal{N}(\theta, \tau^2)$ and suppose there is a variance estimate $s^2 \sim \tau^2 \chi^2(\nu)/\nu$ with a large number $\nu$ of degrees of freedom, independent of $\hat{\theta}$. We choose our units so that the effect size $\theta = 1$. We reject the null hypothesis $\theta = 0$ at level $\alpha \in (0,1)$ if

$$\frac{\hat{\theta}^2}{s^2} = \frac{(\hat{\theta}/\tau)^2}{(s/\tau)^2}$$

exceeds the $1 - \alpha$ quantile of the $F$ distribution with 1 numerator and $\nu$ denominator degrees of freedom. Since $\nu$ is large, the test asymptotically rejects $H_0$ when $\hat{\theta}/\tau$ exceeds the $1 - \alpha$ quantile of the $\chi^2(1)$ distribution.

The independence of $s^2$ from $\hat{\theta}$ won’t always hold. However, we get the same result from $\hat{\theta}$ being asymptotically $\mathcal{N}(\theta, \tau^2)$ and $s^2$ converging in distribution to $\tau^2$.

To study the alternative hypothesis $\theta \neq 0$ it is convenient to scale the measurement units of the problem so that $\theta = 1$. Then $Z \equiv \hat{\theta}/\tau \sim \mathcal{N}(1/\tau, 1)$ and we reject $H_0$ if $Z^2 \sim \chi^2(1)(\tau^{-2}) \geq \chi^2(1)_{\alpha}$ where $\chi^2$ denotes a noncentral $\chi^2$ distribution.

For a given significance level $\alpha \in (0,1)$ and power $1 - \beta \in (\alpha, 1)$, we can solve for the standard error $\tau$. Some R code to do this is in Figure 1. For $\alpha = 0.05$ and $1 - \beta = 0.06$ we find that $\tau = 3.394507$. The point of all those digits in $\tau$ is that we can plug this value in to the \texttt{retrodesign} R function from Gelman and Carlin (2014) and verify that it yields power 0.06 when $\alpha = 0.05$. See Figure 2.
> retrodesign
function(A, s, alpha=.05, df=Inf, n.sims=10000){
z <- qt(1-alpha/2, df)
p.hi <- 1 - pt(z-A/s, df)
p.lo <- pt(-z-A/s, df)
power <- p.hi + p.lo
typeS <- p.lo/power
estimate <- A + s*rt(n.sims,df)
significant <- abs(estimate) > s*z
exaggeration <- mean(abs(estimate)[significant])/A
return(list(power=power, typeS=typeS, exaggeration=exaggeration))
}

> set.seed(1);retrodesign(1,3.394507)
$power
[1] 0.06

$typeS
[1] 0.2013426

$exaggeration
[1] 7.978919

Figure 2: R code from Gelman and Carlin (2014) to verify that standard error \( \tau = 3.394507 \) and effect size \( \theta = 1 \) yields power 0.06 at \( \alpha = 0.05 \). The random seed is set to make the results more reproducible.

The figure in Gelman (2014) (as of October 22, 2016) did not match these results. It had a wrong sign probability of 24% (versus about 20% in Figure 2) and a minimum exaggeration factor of 9 (versus an average of about 8). It appears that the figure in the blog is based on slightly different numbers.

For \( \alpha = 0.05 \) and power 0.06, using \( \tau \) from the code in Figure 1 leads to a critical value of \( \Phi^{-1}(0.975) \times \tau \approx 6.65 \). Therefore any significant discovery must overestimate \( \theta \) by at least 6.65-fold. The average value of \( |\hat{\theta}/\theta| \) given \( \alpha = 0.05 \) and power 0.06 is 8.01 and the probability of a sign error (given rejection of \( H_0 \)) is about 20%. These values closely match the ones from Figure 2 making it more likely that the blog figure is for some power level close to but not exactly 0.06.

3 Error versus power

Here we consider the prevalence of sign errors when \( \alpha = 0.05 \). We vary the power from the low level of 0.06 to the high level 0.80 that is often used as a design goal. Figure 4 shows the conditional probability of a sign error given
Figure 3: The curve shows the $\mathcal{N}(1, \tau^2)$ density of $\hat{\theta}$ when a test of $H_0 : \theta = 0$ at level 0.05 has power 0.06. The central vertical line is at the true value $\theta = 1$. The other vertical lines are at $\hat{\theta} = \pm 6.65$. $H_0$ is only rejected when $|\hat{\theta}| \geq 6.65$. About 20% of rejections have the wrong sign.

that $H_0$ is rejected. It becomes negligible at power greater than about 30%.

Next we consider magnitude errors in $\hat{\theta}$. The light curve in Figure 5 shows the minimal value of $|\hat{\theta}/\theta|$ for which $H_0$ can be rejected at level 0.05. Whenever the power is below 0.5 this minimal ratio is above 1. When the power exceeds 50% then $H_0$ can be rejected with either an exaggerated magnitude or an underestimated magnitude. The darker curve in Figure 5 shows the expected value of $|\hat{\theta}/\theta|$ given that $H_0$ has been rejected, versus the power of the test. Even at 50% power the average exaggeration can be as high as 1.4 fold.

4 Confidence in the sign

If the confidence interval for $\theta$ just barely excludes zero then we should be more cautious about the sign of $\theta$ than if the distance from $\theta$ to 0 is a large multiple of the confidence interval width.

To illustrate, suppose that we reject $H_0$ when $|\hat{\theta}| \geq \Phi^{-1}(0.975)\tau \approx 1.96\tau$. Suppose further that we only declare the sign of $\theta$ to match that of $\hat{\theta}$ when the center $\hat{\theta}$ of the confidence interval is at least $2 \times 1.96\tau$ away from 0. That is, we require the separation between the confidence interval and 0 to be at least half of the confidence interval width. Under $H_0$ such separation only happens with probability $2\Phi(-3.92) \approx 8.85 \times 10^{-5}$. Our chance of wrongly declaring the sign
Figure 4: The figure shows how the probability of a sign error, conditional on rejecting $H_0$, depends on power.

Figure 5: The horizontal axis is the power of a test at level $\alpha = 0.05$. The solid curve is the expected value of $|\hat{\theta}/\theta|$ given that $H_0$ has been rejected. The lighter curve is the minimum value of $|\theta/\theta|$ for which $H_0$ could be rejected.
given that we have rejected \( H_0 \) is now at most

\[
8.85 \times 10^{-5}/0.05 \doteq 0.0018.
\]

This estimate 0.0018 assumes that every time we reject at \( \alpha = 0.05 \) but fail to reject at \( \alpha' = 2\Phi^{-1}(-3.92) \) implies a sign error. The fraction of sign errors is at most 0.5 in the Gaussian setting and is only about 20\% when \( \alpha = .05 \) and the power is 0.06. Therefore a more realistic, yet still conservative estimate of the conditional probability of a sign error given that we have rejected \( H_0 \) and the confidence interval endpoint is at \( \hat{\theta}/2 \) is \( 0.0009 \), just under one in 1000.

Now suppose that \( H_0 \) is rejected at primary significance level \( \alpha_1 \). Next we stipulate a secondary significance level \( \alpha_2 < \alpha_1 \). Let \( R_1 \) be a binary variables taking the value 1 if we reject \( H_0 \) at \( \alpha_1 \) and let \( R_2 \) be the binary variable for \( \alpha_2 \). The probability that we make a sign error under this rule given that we have rejected \( H_0 \) is

\[
\Pr(\text{sign}(\hat{\theta}) \neq \text{sign}(\theta)) \leq \frac{\Pr(R_1 = 1)}{2} \Pr(R_2 = 1) = \frac{1}{2} \frac{\alpha_2}{\alpha_1}.
\]

Given a target conditional sign error of \( \alpha_S \) we use \( \alpha_2 = 2\alpha_1\alpha_S \). For instance, if we want a type I error rate of \( \alpha_1 = 0.01 \) and a conditional sign error rate of at most 0.01, then we take \( \alpha_2 = 2 \times 10^{-4} = 1/5000 \).

Suppose for example that we want to test \( H_0 \) at level \( \alpha = 0.05 \) but we are very averse to sign errors and want \( \alpha_S = 0.001 \). That is at most one in one thousand rejections of \( H_0 \) should come with a sign wrongly declared by the test at level \( \alpha_2 = 0.0001 \). Figure 6 shows both the power to reject \( H_0 \) and the power to identify \( \text{sign}(\hat{\theta}) \) as a function of \( |\theta|/\tau \). It also depicts more lenient sign conditions given by \( \alpha_S = 0.01 \) and \( \alpha_S = 0.1 \). Ordinarily \( \tau = \sigma/\sqrt{n} \) where \( \sigma \) is an asymptotic standard deviation. There is a range of effect sizes in which rejections of \( H_0 \) without decisions on \( \text{sign}(\theta) \) will be common. But once \( |\theta|/\sqrt{n} \geq 5\sigma \) or so, there is very high power that \( H_0 \) will be rejected and a sign will be identified. Also, the hypotheses in limbo with \( H_0 \) rejected but no sign determination are the ones for which obtaining additional data may be most valuable.

In high throughput settings, we might make a very large number of primary hypothesis tests at level \( \alpha_1 \). For each one that is rejected, there is then a smallest \( \alpha_2 \leq \alpha_1 \) at which \( H_0 \) is also rejected. Call this value \( p_2 \). Then the smallest \( \alpha_S \) for which we would have found the sign significant is \( p_S = p_2/(2\alpha_1) \). Note that \( p_S \) depends on the chosen value \( \alpha_1 \), which should then be regarded as a fixed pre-specified choice, and not as a function of the observed data.

## 5 One-sided tests

Here we investigate the possibility of sign errors in one-tailed tests. Those tests are sometimes justified by an assumption that the direction of the effect
is certain a priori. Then sign errors are not possible, unless our certain opinion is wrong. The a priori uncertainty about the sign of $\theta$ must however be small compared to the critical level $\alpha$ in use, and that is a very strong assumption to work under.

A second justification is that sometimes only one direction is consequential. An opinion about one direction being inconsequential might also be mistaken, but for sake of argument we work with it. In this situation we might well make a sign error by rejecting $H_0$. Suppose that $\hat{\theta} \sim \mathcal{N}(1, \tau^2)$, so that $\hat{\theta}/\tau \sim \mathcal{N}(1/\tau, 1)$. The true effect is positive but we might be testing for a (consequential) negative effect.

A one-tailed test at level $\alpha$ in the negative direction would reject $H_0$ if $\hat{\theta}/\tau \leq \Phi^{-1}(\alpha)$. This happens with probability $\Phi(\Phi^{-1}(\alpha) - 1/\tau)$.

Now we revisit the case of 6% power for a two-tailed test at $\alpha = 0.05$. Then $\tau \approx 3.39$ and the sign error is 0.20. The wrong direction one-tailed test at level 0.05 will reject $H_0$ with probability $\Phi(\Phi^{-1}(0.05) - 1/\tau) \approx 0.026$.

The chance of a wrong sign rejection is actually larger than $\alpha/2$ here. That is not to say the conditional probability of a rejection being wrong is over 50%. Indeed in this setting any rejection at all is a sign error. What it does mean is that in this low power setting, the probability of a sign error is not small compared to $\alpha$. As a result, using one-tailed tests does not correct the problem of sign errors in low power settings.
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References


Additional R code

The code here was used to compute the expected value of $\hat{\theta}$ conditionally on $H_0$ being rejected and on sign($\hat{\theta}$).

```r
exag = function(alpha=.05,powr=.06){
  # Compute exaggeration factors
  critse = tau(alpha,powr) # Critical Standard Error
  critz = critse*qnorm(1-alpha/2)
  posmean = gauscondmean(mu=1,sigma=critse,A=critz, B=Inf, n=10^5 )
  negmean = gauscondmean(mu=1,sigma=critse,B=-critz, n=10^5 )
  typeS = pnorm(-critz,1,critse)/powr
  exager = posmean*(1-typeS)+abs(negmean)*typeS
  list(critse=critse,critz=critz,exager=exager,posmean=posmean,negmean=negmean,typeS=typeS)
}

stdgauscondmean = function(A=-Inf, B=Inf, n=10^5, plot=FALSE ){
  # Average of N( 0, 1 ) over interval (A,B)
  u = ((1:n)-0.5)/n
  u = pnorm(A) + (pnorm(B)-pnorm(A)) * u
  z = qnorm(u)
  if( plot )hist(z,50) # for testing/debug
  mean(z)
}

gauscondmean = function(mu=0,sigma=1,A=-Inf, B=Inf, n=10^5 ){
  # Average of N( mu, sigma^-2 ) over interval (A,B)
  mu + abs(sigma)*stdgauscondmean((A-mu)/abs(sigma),(B-mu)/abs(sigma),n)
}
```

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