Effective dimension of weighted Sobolev spaces:
non-periodic case

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Abstract
This paper considers two notions of effective dimension for quadrature in weighted Sobolev spaces. A space has effective dimension \( s \) in the truncation sense if a ball in that space just large enough to contain a function of unit variance does not contain any functions with more than \( \varepsilon \) variance attributable to ANOVA components of indices past \( s \). A similar truncation sense notion replaces index \( s \) by cardinality \( s \). Some Poincaré type inequalities are used to bound variance components by multiples of the Sobolev space’s squared norm and these in turn provide bounds on effective dimension. Very low effective dimension in the superposition sense holds for some spaces defined by product weights in which quadrature is strongly tractable. Surprisingly, even spaces where all subset weights are equal, regardless of their cardinality or included indices, have low superposition dimension in this sense. A previous paper by the author required the integrands to have periodic mixed partial derivatives. This paper removes that condition.

1 Introduction
This paper gives upper bounds for the effective dimension of certain weighted Sobolev spaces of functions on \([0, 1]^d\). The notion of effective dimension used here is the one from Owen (2014). That work was based on a Fourier expansion and it required a periodicity condition on certain partial derivatives of the functions \( f \). In this paper, no periodicity assumption is required.

The Sobolev spaces we consider are used as models for certain numerical integration problems. There we seek \( \mu = \int_{[0, 1]^d} f(x) \, dx \) where \( 1 \leq d < \infty \), and \( d \) might be large. For large \( d \), integration becomes quite hard for worst case integrands, even smooth ones, yielding a curse of dimensionality described by Bakhvalov (1959). Quasi-Monte Carlo (QMC) integration (Niederreiter, 1992; Dick and Pillichshammer, 2010), often succeeds in high dimensions despite the curse. This can be explained by the functions having less than full \( d \)-dimensional complexity. Weighted Sobolev spaces provide one model for such reduced complexity. This paper translates the weights defining those spaces into bounds.
on certain $L^2$ norms quantifying the notion that integrands in those spaces are ‘effectively’ $s$ dimensional where $s$ might be much less than $d$.

The functions we consider have a partial derivative taken once with respect to each of $d$ coordinates, and moreover, that partial derivative is a continuous function on $[0,1]^d$. That is sufficient smoothness to place them in the weighted Sobolev spaces mentioned above. Such weighted spaces have been used to model settings in which higher order interactions (Hickernell, 1996), or successive dimensions, or both (Sloan and Woźniakowski, 1998) are less important. When these high order or high index components decay quickly enough, the result is a set of functions that evades the curse of dimensionality established by Bakhvalov (1959).

The contribution of various high order or high index parts of an integrand to Monte Carlo (MC) variance can be quantified through the analysis of variance (ANOVA) decomposition defined below. Suppose that an integrand $f$ has less than one percent of its variance coming from high dimensional components, and that another quadrature method proves to be far more accurate than plain MC. That improvement cannot be attributed to better handling of the high dimensional parts, because they caused at most one percent of the squared error for Monte Carlo. The improvement must have come from superior handling of the low dimensional aspects of $f$.

In this paper we investigate some senses in which a space of functions is of low effective dimension. We look at some weighted Sobolev norms (and some semi-norms) and define measures of the extent to which balls in those normed spaces are dominated by their low dimensional parts with respect to an ANOVA decomposition. We select a ball in which the worst case Monte Carlo variance is unity, and then consider whether any integrand in that ball has meaningfully large variance coming from its high dimensional components. If not, then the space itself is said to have low effective dimension.

An outline of the paper is as follows. Section 2 gives our notation, introducing weighted Sobolev spaces with some conditions on the weights and the ANOVA decomposition of $L^2([0,1]^d)$. Section 3 introduces some Poincaré type inequalities that we use to lower bound the Sobolev squared norms in terms of ANOVA components. Section 4 defines what it means for a space of functions to have a given effective dimension. Given a ball of fixed radius in that space, if no function inside that ball has a meaningfully large high dimensional ANOVA component, then the ball itself is said to be of low effective dimension. Section 5 gives upper bounds on the effective dimension of a space in terms of its weights under a monotonicity condition that gives smaller weights to higher order and higher indexed subsets of variables. Explicit effective dimension bounds are worked out and tabulated. One surprise in this work is that giving every subset equal weight, regardless of its cardinality or the size of the indices it contains, still leads to modest superposition dimension. Section 6 has some conclusions.

To finish this section, we survey some literature. Poincaré inequalities have been used in the global sensitivity analysis (GSA) literature to bound Sobol’ indices. See for instance Sobol’ and Kucherenko (2009, 2010) and Lamboni et al. (2012). Roustant et al. (2014) use Poincaré inequalities based on mixed partial
derivatives to bound some superset variable importance measures in GSA.

Wang and Fang (2003) defined a different notion of effective dimension for spaces, through what they call a typical function in the space, derived from the weights that define the space. Their formulation depended on the nominal dimension \( d \) whereas the definition here is unchanged if we append unused dimensions. Their computed dimensions are similar to but not identical to the ones here.

There have been several recent papers on truncation dimension for infinite dimensional spaces. If \( f \) is defined on an infinite dimensional space we can usually only prescribe some finite number \( k \) of its input values. All other inputs may be viewed as taking a default value, such as 0. Kritzer et al. (2016) define the truncation dimension of a weighted space through an anchored decomposition. In their definition, the space has effective dimension \( k \) if the difference between an estimate computed using just \( k \) inputs and a computation using all inputs is below some small multiple of the norm of that weighted space. The bound still holds when the ‘all inputs’ computation is hypothetical, as it could be for infinite dimensional problems. They obtain effective dimensions for Banach spaces. Hinrichs et al. (2017) extend the results to general linear operators. Kritzer et al. (2017) establish bounds on embeddings between anchored spaces and the ANOVA spaces we consider here.

2 Notation

The indices of \( x \in [0,1]^d \) are \( j \in \{1,2,...,d\} \equiv 1:d \). For \( u \subseteq 1:d \), we use \(|u|\) for its cardinality and \(-u\) for its complement with respect to 1:d. For \( u \subseteq 1:d \), the point \( x_u \in [0,1]|^u \) consists of the components \( x_j \) for \( j \in u \). Given \( x, z \in [0,1]^d \) the hybrid point \( y = x_u : z_{-u} \) has \( y_j = x_j \) for \( j \in u \) and \( y_j = z_j \) for \( j \notin u \). We use \( 0 = (0,0,...) \) for the \( d \)-dimensional origin.

The differential \( dx_u \) is \( \prod_{j \in u} dx_j \). Similarly, \( \partial^u f \) denotes \( \partial^{|u|} f / \prod_{j \in u} \partial x_j \), and by convention \( \partial^\emptyset f = f \). The functions we consider belong to 

\[ \mathcal{F} = \{ f : [0,1]^d \to \mathbb{R} \mid \partial^{1:d} f \text{ is continuous on } [0,1]^d \}. \]

Continuity here allows the partial derivatives to be taken in any order and it allows some interchanges of order between differentiation and integration.

2.1 Weighted spaces

Let \( \gamma_u > 0 \) for all \( u \subseteq 1:d \) and let \( \gamma \) comprise all of those choices for \( \gamma_u \). We use an unanchored norm defined by

\[
\| f \|_\gamma^2 = \sum_{u \subseteq 1:d} \frac{1}{\gamma_u} \int_{[0,1]^{|u|}} \left( \int_{[0,1]^{d-|u|}} \partial^u f(x) \, dx_{-u} \right)^2 \, dx_u.
\]  

See Dick et al. (2013) for background on this and a related anchored norm. This norm is finite for every choice of \( \gamma \) and all \( f \in \mathcal{F} \). Some of our results allow
a semi-norm instead found by dropping $u = \emptyset$ from the sum, or equivalently, taking $\gamma_\emptyset = \infty$. More generally, the $u = \emptyset$ term is $\mu^2 / \gamma_\emptyset$ where $\mu = \int f(x) \, dx$. There are now many efficient methods of constructing quasi-Monte Carlo point sets for weighted spaces. See Sloan et al. (2002), Nuyens and Cools (2006a) and Nuyens and Cools (2006b).

Numerous choices of weights are given in the survey Dick et al. (2013), and a comprehensive treatment is available in Novak and Woźniakowski (2010). Sloan and Woźniakowski (1998) use product weights $\gamma_u = \prod_{j \in u} \gamma_j$ including $\gamma_\emptyset = 1$. Typically $1 = \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_j \geq \gamma_{j+1} \geq \cdots > 0$. Hickernell (1996) uses weights $\gamma_u = |u|^\alpha$ for some $\gamma \in (0, 1)$. Such weights are commonly called order weights. The more general order weights of Dick et al. (2006) take the form $\gamma_u = \Gamma |u|$ where $\Gamma_r \geq 0$ is a nonincreasing function of $r$. The case with $\Gamma_r = 0$ for all $r \geq r_0$, known as finite-order weights, was studied by Sloan et al. (2004). However Sloan (2007) points out a danger from this choice. We will suppose that $\Gamma_r > 0$. Dick et al. (2006) also consider completely general weights $\gamma_u$ but Dick et al. (2013) note that such generality sharply raises the costs of using those weights to design an algorithm. Product and order weights, known as POD weights, defined by $\gamma_u = |u|^\alpha \times \prod_{j \in u} j^{-\beta}$ have proved useful in QMC based algorithms for solving PDEs with random coefficients (Kuo et al., 2012b). One useful choice has $\gamma_u = (|u|!)^\alpha \times \prod_{j \in u} j^{-\beta}$ where $0 < \alpha < \beta$. Note that with this choice, $\Gamma_{|u|} = (|u|!)^\alpha$ is increasing in $|u|$.

Higher weights are placed on the more important subsets and other things being equal, subsets with fewer components are considered more important as are subsets containing components with lower indices.

We partially order subsets by $|u|$ and we use a parallel notation $[u] = \max\{j \mid j \in u\}$ to order subsets by their largest element. By convention, $[\emptyset] = 0$. Two frequently satisfied conditions on the weights are:

$$\max\{\gamma_u \mid |u| = s\} = \gamma_{1:s}, \quad \text{and}$$

$$\max\{\gamma_u \mid [u] = s\} = \gamma_{\{s\}}. \quad (2)$$

Most of the widely studied weights satisfy both (2) and (3). The POD weights $\gamma_u = (|u|!)^\alpha \times \prod_{j \in u} j^{-\beta}$ are an exception. They satisfy (2) but not (3). For these weights $\gamma_{u \cup \{1\}} > \gamma_u$ whenever $1 \notin u$.

Together, conditions (2) and (3) imply that $\gamma_u \leq \gamma_{\{1\}}$ for all $u \neq \emptyset$. This also holds for the POD weights described above. Regarding $1/\gamma_u$ as a multiplicative penalty factor, the singleton $\{1\}$ is the ‘least penalized’ index subset, though it may not be uniquely least penalized. Many but not all weights in use satisfy $\gamma_u \geq \gamma_\emptyset$ when $u \subseteq v$.

2.2 ANOVA

This section gives notation for the Hoeffding-Sobol’ ANOVA decomposition of $L^2([0,1]^d)$, which contains every $f \in \mathcal{F}$. There is additional background in Owen (2013). In this decomposition $f(x) = \sum_{u \subseteq 1:d} f_u(x)$ where $f_u$ depends on $x$ only through $x_u$. Many decompositions of this type are possible. The
ANOVA uniquely satisfies \( \int_0^1 f_u(x) \, dx = 0 \) whenever \( j \in u \). For \( u \neq \emptyset \) define the variance component \( \sigma_u^2 = \int_{[0,1]^d} f_u(x)^2 \, dx \), and take \( \sigma_\emptyset^2 = 0 \). Then the variance of \( f \) decomposes as

\[
\sigma^2 = \int_{[0,1]^d} (f(x) - \mu)^2 \, dx = \sum_{u \subseteq 1^d} \sigma_u^2.
\]

3 Some Poincaré type inequalities

Our main tool will be bounds on \( L^2 \) norms based on integrated squared derivatives. This section presents them with some history. These are generally known as Poincaré inequalities. Poincaré worked with integrals over more general domains than \([0,1]\) as well as more general differential operators than those used here.

**Theorem 1.** Let \( f \) be differentiable on the finite interval \((a,b)\) and satisfy \( \int_a^b f(x) \, dx = 0 \). Then

\[
\int_a^b f'(x)^2 \, dx \geq \left( \frac{\pi}{b-a} \right)^2 \int_a^b f(x)^2 \, dx,
\]

and equality is attained for some nonzero \( f \).

**Proof.** This is on pages 295–296 of Stekloff (1901).

The constant \( \pi \) in equation (4) will appear in our formulas. Things would be different if we were to work on the interval \([0,\pi]\) or \([0,2\pi]\). We retain a focus on \([0,1]\) because weighted spaces are almost always defined over \([0,1]^d\).

Theorem 1 is commonly known as Wirtinger’s theorem, though Stekloff’s work was earlier. Sobol’ (1963) has a different proof than Stekloff, based on the calculus of variations. This theorem is often given with an additional condition that \( f(a) = f(b) \) (periodicity), though such a condition is not necessary.

This inequality could be much older than 1901. See Mitroinovic et al. (1991, Chapter 2). The condition \( \int_a^b f(x) \, dx = 0 \) can be removed if \( f(a) = f(b) = 0 \). Equality is attained if \( f(x) \) is a multiple of \( \sin(\pi(x-m)/s) \) for \( m = (a+b)/2 \) and \( s = b-a \).

**Lemma 1.** Let \( f(x) \) defined on \([0,1]^d\) satisfy \( \int_0^1 f(x) \, dx = 0 \) for all \( j = 1,\ldots,d \). If \( d \geq r \geq 0 \), and \( \partial^{1r} f \) exists, then

\[
\int_{[0,1]^d} \left( \frac{\partial^r}{\partial x_1 \cdots \partial x_r} f(x) \right)^2 \, dx \geq \pi^{2r} \int_{[0,1]^d} f(x)^2 \, dx.
\]

**Proof.** This holds for \( r = 0 \) by convention. It holds for \( d = r = 1 \) by Theorem 1. It extends to \( d \geq r \geq 1 \) by induction.
Theorem 2. Let \( f \) be defined on \( [0, 1]^d \) with \( \partial^{1:d} f \) continuous. Furthermore let \( f \) have ANOVA effects \( f_v \) for \( v \subseteq 1:d \) with variance components \( \sigma_v^2 \). Then
\[
\|f\|_2^2 \geq \gamma_0^{-1} \mu^2 + \sum_{u \neq \emptyset} \gamma_u^{-1} \pi^{2|u|} \sigma_u^2. \tag{6}
\]

Proof. Let \( f \) have ANOVA effects \( f_v \) for \( v \subseteq 1:d \). Then for \( u \neq \emptyset \),
\[
\int_{[0,1]^{d-|u|}} \partial^u \sum_{v \subseteq 1:d} f_v(x) \, dx_{-u} = \partial^u \int_{[0,1]^{d-|u|}} \sum_{v \supseteq u} f_v(x) \, dx_{-u} = \partial^u f_u(x_u \cdot 0_{-u}),
\]
because \( f_u \) does not depend on \( x_{-u} \). Next \( \int_{[0,1]^{1:|u|}} (\partial^u f_u(x_u \cdot 0_{-u}))^2 \, dx_u \geq \pi^{2|u|} \sigma_u^2 \) by Lemma 1. The result follows from (1). \( \square \)

4 Effective dimension of a space

There are two main notions of effective dimension for a function (Caflisch et al., 1997). Here we review them in order to motivate a definition for the effective dimension of a space of functions.

A function on \( [0, 1]^d \) has effective dimension \( s \leq d \) in the truncation sense, if it has negligible dependence upon components \( x_j \) for \( j > s \). It has effective dimension \( s \leq d \) in the superposition sense if it has negligible dependence upon interactions among \( s+1 \) or more components. These definitions are quantified in terms of variances which allows us to study when QMC, especially randomized QMC, brings a major improvement over MC.

Let \( f \) have ANOVA effects \( f_u \) with variances \( \sigma_u^2 \) for \( u \subseteq 1:d \). Then \( f \) has effective dimension \( s \) in the superposition sense if \( s \) is the smallest integer with
\[
\sum_{u:|u| \leq s} \sigma_u^2 \geq 0.99 \sigma^2.
\]
It has effective dimension \( s \) in the truncation sense if \( s \) is the smallest integer with
\[
\sum_{u:|u| \leq s} \sigma_u^2 \geq 0.99 \sigma^2.
\]

The arbitrary constant 0.99 may be explained as follows. If we use plain Monte Carlo with \( n \) observations then our variance is \( \sigma^2/n = \sum_u \sigma_u^2/n \). If more than 99% of the variance comes from some subset of effects \( f_u \), then a method such as QMC which can integrate them at a better rate of convergence has a possibility of attaining a 100-fold improvement over MC, even if it is no better than MC for the other effects. Users can reasonably ignore a method that is say twice as fast, due to tradeoffs in implementation difficulty or even familiarity. When a 100-fold improvement is available, it should be harder to ignore.
Now consider balls of real-valued functions on $[0,1]^d$ given by
\[ \mathcal{B}(\gamma, \rho) = \{ f \mid \|f\|_\gamma \leq \rho \} \]
where $\rho > 0$. We pick the radius of the ball to make it just large enough to contain a function of unit variance. That radius is
\[ \rho^* = \rho^*(\gamma) = \inf\{ \rho > 0 \mid \exists f \in \mathcal{B}(\gamma, \rho) \text{ with } \sigma^2(f) = 1 \}. \tag{7} \]
The ball is said to be of effective dimension $s$ in the truncation sense at level $1 - \varepsilon$ (such as $\varepsilon = 0.01$) if $s$ is the smallest integer for which
\[ \sup_{f \in \mathcal{B}(\gamma, \rho^*)} \sum_{u: |u| > s} \sigma_u^2(f) < \varepsilon. \tag{8} \]
That is
\[ \sup_{f \in \mathcal{B}(\gamma, \rho^*)} \sum_{u: |u| > s} \sigma_u^2(f) < \varepsilon \leq \sup_{f \in \mathcal{B}(\gamma, \rho^*)} \sum_{u: |u| \geq s} \sigma_u^2(f). \tag{9} \]
It is of effective dimension $s$ in the superposition sense at level $1 - \varepsilon$, if $s$ is the smallest integer for which
\[ \sup_{f \in \mathcal{B}(\gamma, \rho^*)} \sum_{u: |u| > s} \sigma_u^2(f) < \varepsilon. \tag{10} \]
If the ball is of low effective dimension, then no function in it contains any integrands with non-negligible high order components, where order is quantified by $|u|$ or $|u|$. Here $\varepsilon = 0.01$ corresponds to the usual choice, but we will want to see how effective dimension depends on $\varepsilon$.

Normalizing to variance one is interpretable, but not really necessary. We can work with ratios of norms. An equivalent condition to (8) is that $s$ is the smallest integer for which
\[ \sup_{f: 0 < \|f\|_\gamma < \infty} \frac{\sum_{|u| \geq s+1} \sigma_u^2(f)}{\|f\|_\gamma^2} < \varepsilon, \]
and so the definition for the ball $\mathcal{B}(\gamma, \rho^*)$ is really a property of the weights $\gamma$. 

**Proposition 1.** Let weights $\gamma_u$ be such that $\gamma_{\{1\}} \pi^{1-2} \geq \gamma_u \pi^{-2|u|}$ for all non-empty $u \subseteq 1:d$. Then the smallest $\rho$ for which $\mathcal{B}(\gamma, \rho)$ contains a function of variance 1 is $\rho^*(\gamma) = \pi(\gamma_{\{1\}})^{-1/2}$. 

**Proof.** Let $f$ have $\sigma^2(f) = 1$ and suppose that $\rho < \rho^*$. Then
\[ \|f\|_\gamma^2 \geq \mu^2 \gamma_{\emptyset}^{-1} + \min_{u \neq \emptyset} \gamma_u^{-1} \pi^{2|u|} \geq \gamma_{\{1\}}^{-1} \pi^2 = (\rho^*)^2 > \rho^2, \]
so $f \not\in \mathcal{B}(\gamma, \rho)$. If $f(x) = \sqrt{2}\sin(\pi(x_1 - 1/2))$, then $\sigma^2(f) = 1$ and $f \in \mathcal{B}(\gamma, \rho^*)$. □
Weights with $\gamma_u \leq \gamma_{\{1\}}$ automatically satisfy the condition in Proposition 1.

**Proposition 2.** For weights $\gamma$, let $\rho^*(\gamma)$ be defined by (7), let $u \subseteq 1:d$ be non-empty, and let $f \in B(\gamma, \rho^*(\gamma))$. Then

$$\sigma_u^2(f) \leq \rho^*(\gamma)^2 \gamma_u \pi^{-2[u]}.$$

If also $\gamma_{\{1\}} \pi^{-2} \geq \gamma_u \pi^{-2[u]}$, then

$$\sigma_u^2(f) \leq \gamma_{\{1\}} \pi^{-2(u-1)} \gamma_u / \gamma_{\{1\}}.$$

**Proof.** From Theorem 2, $\rho^* \geq \|f\gamma\| \geq \gamma_{\{1\}}^{-1} \pi^{-2[u]} \sigma_u^2$, establishing the first claim. Then the second one follows from Proposition 1. □

## 5 Bounds on effective dimension

**Theorem 3.** Let the weights $\gamma$ satisfy (2). Then their effective dimension in the superposition sense at level $1 - \varepsilon$ is at most

$$\max\{s \geq 1 \mid \gamma_{\{1:s\}} \geq \pi^{2(s-1)} \gamma_{\{1\}} \varepsilon\}. \tag{11}$$

If the weights $\gamma$ satisfy (3), then their effective dimension in the truncation sense at level $1 - \varepsilon$ is at most

$$\max\{s \geq 1 \mid \gamma_{\{s\}} \geq \gamma_{\{1\}} \varepsilon\}. \tag{12}$$

**Proof.** Let $f \in B(\gamma, \rho^*)$ where $\rho^*$ is given by (7). Let $f$ have ANOVA variance components $\sigma_u^2$. Choose an integer $s > 0$. From Proposition 1, Theorem 2 and condition (2),

$$\gamma_{\{1\}} \pi^{-2} = \rho^* \geq \|f\gamma\| \geq \mu^2 \gamma_{\{1\}}^{-1} + \sum_{u \neq \emptyset} \gamma_u^{-1} \pi^{-2[u]} \sigma_u^2 \geq \gamma_{\{1:s\}} \pi^{-2|u|} \sum_{|u| \geq s} \sigma_u^2.$$

Therefore $\sum_{|u| \geq s} \sigma_u^2 \leq \gamma_{\{1:s\}} \pi^{-2(s-1)} / \gamma_{\{1\}}$. If $\gamma_{\{1:s\}} < \gamma_{\{1\}} \pi^{2(s-1)} \varepsilon$, then $\sum_{|u| \geq s} \sigma_u^2 < \varepsilon$ and the effective dimension cannot be as large as $s$. This establishes the superposition bound (11).

For the truncation dimension, we find that $\gamma_{\{1\}} \pi^{-2} \geq \gamma_{\{s\}} \pi^{-2} \sum_{|u| \geq s} \sigma_u^2$, by a similar argument to the one used above, this time using (3). Then (12) follows just as (11) did. □

### 5.1 Tractability and product weights

Here we look at product weights of the form $\gamma_u = \prod_{j \in u} \gamma_j$ for monotone values $\gamma_j \geq \gamma_{j+1}$ for $j \geq 1$, including $\gamma_\emptyset = 1$. Sloan and Woźniakowski (1998) give conditions on the weights for high dimensional quadrature to be tractable, which we define next. They consider a sequence of $d$-dimensional settings in which $d \to \infty$. We will look at their weights restricted to $u \subseteq 1:d$. We draw on the summary of tractability results given by Kuo et al. (2012a).
Suppose that \( f \in B(\gamma, \rho) \), for \( 0 < \rho < \infty \). If we had to pick an \( n = 0 \) point rule for functions in \( B(\gamma, \rho) \) it would be a constant and we minimize worst case error by taking that constant to be 0. Our initial error is then \( \sup_{f \in B(\gamma, \rho)} | \int f(x) \, dx | \). Now let \( n = n_\gamma(\epsilon, d) \) be the smallest integer for which some QMC rule reduces the initial error by a factor of \( \epsilon \). This \( n \) does not depend on our choice \( \rho \).

The problem of quadrature is defined to be tractable if there exist points \( x_i \) with \( n_\gamma(\epsilon, d) \leq Cd^{\eta} \) for non-negative constants \( C, \eta \) and \( p \). If \( q = 0 \) is possible, then the cost \( n \) can be taken independent of dimension \( d \), and the problem is then said to be strongly tractable. The problem of quadrature is strongly tractable if \( \sum_{j=1}^\infty \gamma_j < \infty \) (Sloan and Woźniakowski, 1998). Their result was nonconstructive and it had \( p = 2 \), comparable to plain Monte Carlo. Hickernell and Woźniakowski (2000) gave an improved non-constructive proof showing errors \( n^{-1+\delta} \) are possible (so \( p = 1/(1-\delta) \)) if

\[
\sum_{j=1}^\infty \gamma_j^{1/2} < \infty. \tag{13}
\]

Of special interest is the case \( \eta = 2 \). For any \( \eta > 2 \), (13) would hold and then there exist QMC points with worst case errors that decrease at the rate \( O(n^{-1+\delta}) \) for any \( \delta > 0 \). A second interesting value is \( \eta = 1 \). For any \( \eta > 1 \), strong tractability holds but not at a better rate than Monte Carlo provides.

5.2 Effective dimension

We consider weight factors of the form \( \gamma_j = j^{-\eta} \) for various \( \eta \). We always have \( \gamma_1 = 1 \) as Sloan and Woźniakowski (1998) do. Taking \( \gamma_1 > 1 \) would yield \( \gamma_{11} > \gamma_2 \) which seems unreasonable because constant functions can be integrated without error for \( n \geq 1 \).

From Theorem 3 we see that the truncation dimension is at most

\[
\max\{s \geq 1 \mid s^{-\eta} \geq \varepsilon\}
\]

and the superposition dimension is at most

\[
\max\{s \geq 1 \mid (s!)^{-\eta} \geq \pi^{2(s-1)} \varepsilon\}.
\]

Both of these effective dimensions are non-increasing with \( \eta \).

Recall that the choice \( \eta = 2 \) is just barely too small for strong tractability at better than the MC rate. The effective dimension for \( \eta > 2 \) will be at most that of the case \( \eta = 2 \). Similarly \( \eta = 1 \) is just barely too small for strong tractability at the MC rate. We also look at the case \( \eta = 0 \) with equal weights \( \gamma_u = 1 \) for all \( u \subseteq 1:d \). Integration is not tractable for such weights.

The effective dimensions in both the truncation and superposition senses for product weights with \( \eta \in \{0, 1, 2\} \) are given in Table 1. These values are identical to those in Owen (2014) under a periodicity constraint except that the
Table 1: Upper bounds on effective dimension for product weights defined by \( \gamma_j = j^{-\eta} \).

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( \eta = 2 )</th>
<th>( \eta = 1 )</th>
<th>( \eta = 0 )</th>
<th>( \eta = 2 )</th>
<th>( \eta = 1 )</th>
<th>( \eta = 0 )</th>
</tr>
</thead>
<tbody>
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<td>3</td>
<td>10</td>
<td>( \infty )</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0.01</td>
<td>9</td>
<td>100</td>
<td>( \infty )</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0.001</td>
<td>31</td>
<td>1000</td>
<td>( \infty )</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.0001</td>
<td>99</td>
<td>10000</td>
<td>( \infty )</td>
<td>3</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

truncation dimension for \( \varepsilon = 10^{-4} \) given there is 101 instead of 100. In this case, equality holds in (12) for \( \eta = 2 \) and \( s = 100 \).

For weights given by \( \eta = 2 \) and using \( \varepsilon = 0.01 \) we find that the truncation dimension is at most 9 and the superposition dimension is at most 2. When \( \eta = 0 \), the truncation dimension is unbounded. For instance, an integrand of variance one depending only on \( x_j \) would be inside the unit ball. Since there is no a priori upper bound on \( j \) we get \( \infty \) for the truncation dimension. It is surprising that the superposition dimension is not very large for \( \eta = 0 \). For \( \eta = 0 \), all of the weights are \( \gamma_u = 1 \). That would seem intuitively to describe equally important subsets but instead it implies some dominance of low cardinality subsets.

5.3 Most important interactions

For \( \eta = 2 \) and \( \varepsilon = 0.01 \), we find that the only large variance components involve only one or two of the first 9 input variables. We can also investigate which of the two factor interactions could be large.

Using Proposition 2 we can bound \( \sigma_u^2 \). For product weights with \( \gamma_j = j^{-2} \) we get \( \sigma_u^2 \{1, 2\} \leq \pi^{-2}/4 \approx 0.025 \) and \( \sigma_u^2 \{1, 3\} \leq \pi^{-2}/9 \approx 0.011 \). The other two variable interactions have bounds below 0.01 as do interactions of order three and up.

If we lower the threshold to \( \varepsilon = 0.001 \), then \( \sigma_u^2 \) for \( u = \{1, j\} \) and \( 2 \leq j \leq 10 \) are potentially this large as are those for \( u = \{2, j\} \) for \( j = 3, 4, 5 \), but \( \sigma_u^2 < \varepsilon \) for all \( |u| \geq 3 \).

For \( \eta = 1 \), the sets \( u \) where the upper bound on \( \sigma_u^2 \) is below \( \varepsilon = 0.01 \) are singletons \( \{1\} \) to \( \{100\} \) and the same two factor interactions which meet the \( \varepsilon = 0.001 \) criterion for \( \eta = 2 \).

These or very similar subsets were found independently to be important in anchored spaces by Greg Wasilkowski who presented them at the SAMSI workshop on QMC.

6 Conclusions

The effectiveness of QMC sampling on nominally high dimensional integrands can be explained in part by those specific integrands having a low effective
dimension (Caflisch et al., 1997) as measured by ANOVA. It is only partial because ANOVA components are not necessarily smooth enough for QMC to be beneficial. It is however common for the process by which ANOVA components are defined to make the low order components smooth (Griebel et al., 2010).

This paper considers effective dimension of weighted Sobolev spaces without requiring periodicity of the integrands. Some weighted Sobolev spaces describe families of integrands over which QMC has uniformly good performance. A ball in such a space just barely large enough to contain an integrand of unit variance, will contain no integrands with meaningfully high dimensional or high index variance components. Thus algorithms for integration in these settings must focus on certain low dimensional aspects of the input space.

It is surprising that weighted spaces with all \( \gamma_u = 1 \) (which does not allow tractability) leads to spaces with modest superposition truncation dimension. In this sense, \( \gamma_u = 1 \) is not a model for a situation where all subsets of variables are equally important. Perhaps unit weights on \([0, \pi]^d\) would describe a setting where all subsets of variables are equally important.

Up to this point, we have emphasized functions of low effective dimension. It is important to consider what happens if the integrand \( f \) at hand is not dominated by its low dimensional components. If \( f \) is in the unit ball in one of the weighted Sobolev spaces, then a modestly large \( n \) can be found which will yield an integration error smaller than \( \epsilon \) for this \( f \) and all other integrands in that ball. Even if \( f \) is not in that ball, \( \bar{f} = f/\|f\|_\gamma \) is in that ball, and we can be sure of an error below \( \epsilon \) for \( \bar{f} \). Using MC and QMC methods, this means we have an error below \( \epsilon \|f\|_\gamma \) for \( f \). If \( \|f\|_\gamma \) is very large then we need a very small \( \epsilon \) to compensate. A function such as \( f_d = \prod_{j=1}^d (x_j - 1/2) \) makes a good test case. For product weights \( \|f_d\| = (d!)^{\eta/2} \), and so for good results we would need \( \epsilon \) to be comparable to \( (d!)^{-\eta/2} \), which then requires \( n \) to be a power of \( d! \).

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