Supplementary material for: Efficient moment calculations for variance components in large unbalanced crossed random effects models

Katelyn Gao
Stanford Univesity

Art B. Owen
Stanford University

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Abstract

This is a supplementary document containing proofs for some results in the main document. The section numbers continue where that document left off. Some contextual material is repeated for clarity. Also as this is a supplementary document, material that is traditionally left out as being 'tedious algebra' is included in full detail, making the numerous steps easier to follow and check.

10 Partially observed random effects model

The random effects model is

$$Y_{ij} = \mu + a_i + b_j + e_{ij}, \quad i, j \in \mathbb{N} \tag{100}$$

for $a_i \overset{\text{iid}}{\sim} F_a$, $b_j \overset{\text{iid}}{\sim} F_b$ and $e_{ij} \overset{\text{iid}}{\sim} F_e$ independent of each other. These random variables have mean 0, variances σ_A^2, σ_B^2, σ_E^2 and kurtoses κ_A, κ_B, κ_E, respectively. We will not need their skewnesses.

We use letters i, i', r, r' to index rows. Letters j, j', s, s' are used for columns. In internet applications, the actual indices may be people rating items, items being rated, cookies, URLs, IP addresses, query strings, image identifiers and so on. We simplify the index set to $\mathbb{N}$ for notational convenience. One feature of these variables is that we fully expect future data to bring hitherto unseen levels. That is why a countable index set is appropriate.

We will want to estimate σ_A^2, σ_B^2, σ_E^2 and get a formula for the variance of those estimates. Many, perhaps most, of the Y_{ij} values are missing. Here we assume that the missingness is not informative. For further discussion see Section 7.1 of the main document.

The variable $Z_{ij} \in \{0, 1\}$ takes the value 1 if Y_{ij} is available and 0 otherwise. The total sample size is $N = \sum_{ij} Z_{ij}$. We assume that $1 \leqslant N < \infty$. We also need $N_{i\bullet} = \sum_j Z_{ij}$ and $N_{\bullet j} = \sum_i Z_{ij}$. The number of unique observed rows and columns are, respectively,

$$R \equiv \sum_i 1_{N_{i\bullet}>0}, \quad \text{and} \quad C \equiv \sum_j 1_{N_{\bullet j}>0}.$$

In the sum above, only finitely many summands are nonzero. When we sum over i, i', r, r', the sum is over the set $\{i \mid N_{i\bullet} > 0\}$. Similarly sums over column indices j, j', s, s' are over the set $\{j \mid N_{\bullet j} > 0\}$. These ranges are what one would naturally get in a pass over data logs showing all records.

We frequently need the number of columns jointly observed in two rows such as i and i'. This is $\sum_j Z_{ij} Z_{i'j} = (ZZ^{\mathsf{T}})_{ii'}$. Similarly, columns j and j' are jointly observed in $\sum_i Z_{ij} Z_{ij'} = (Z^{\mathsf{T}} Z)_{jj'}$ rows.

The matrix Z encodes several different measurement regimes as special cases. These include crossed designs, nested designs and IID sampling, as follows. A crossed design with an $R \times C$ matrix of completely observed data can be represented via $Z_{ij} = 1_{1 \leqslant i \leqslant R} 1_{1 \leqslant j \leqslant C}$. If $\max_i N_{i\bullet} = 1$ and $\max_j N_{\bullet j} > 1$ then the data have a nested structure, with $N_{\bullet j}$ distinct rows in column j and $(Z^{\mathsf{T}} Z)_{jj'} = 0$ for $j \neq j'$. Similarly

$\max_j N_{\bullet j} = 1$ with $\max_i N_{i\bullet} > 1$ yields columns nested in rows. If $\max_i N_{i\bullet} = \max_j N_{\bullet j} = 1$ then we have N IID observations.

We note some identities:

$$\sum_{ir} (ZZ^{\mathsf{T}})_{ir} = \sum_{ijr} Z_{ij} Z_{rj} = \sum_j N_{\bullet j}^2, \quad \text{and} \tag{101}$$

$$\sum_{ir} N_{i\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir} = \sum_{ijr} N_{i\bullet}^{-1} Z_{ij} Z_{rj} = \sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{\bullet j}. \tag{102}$$

We need some notation for equality among index sets. The notation $1_{ij=rs}$ means $1_{i=r}1_{j=s}$. It is different from $1_{\{i,j\}=\{r,s\}}$ which we also use. Additionally, $1_{ij\neq rs}$ means $1 - 1_{ij=rs}$.

11 Weighted U statistics

We will work with weighted U-statistics

$$\begin{aligned} U_a &= \frac{1}{2} \sum_{ijj'} u_i Z_{ij} Z_{ij'} (Y_{ij} - Y_{ij'})^2 \\ U_b &= \frac{1}{2} \sum_{iji'} v_j Z_{ij} Z_{i'j} (Y_{ij} - Y_{i'j})^2, \quad \text{and} \\ U_e &= \frac{1}{2} \sum_{iji'j'} w_{ij} Z_{ij} Z_{i'j'} (Y_{ij} - Y_{i'j'})^2, \end{aligned}$$

for weights u_i, v_j and w_{ij} chosen below.

We can write $U_a = \sum_i u_i N_{i\bullet}(N_{i\bullet} - 1)s_{i\bullet}^2$ where $s_{i\bullet}^2$ is an unbiased estimate of $\sigma_B^2 + \sigma_E^2$ from within any row i with $N_{i\bullet} \geqslant 2$. Under our model the values in row i are IID with mean $\mu + a_i$ and variance $\sigma_B^2 + \sigma_E^2$, and so

$$\mathrm{Var}(s_{i\bullet}^2) = (\sigma_B^2 + \sigma_E^2)^2 \Bigl(\frac{2}{N_{i\bullet} - 1} + \frac{\kappa(b_j + e_{ij})}{N_{i\bullet}} \Bigr)$$

where $\kappa(b_j + e_{ij}) = (\kappa_B \sigma_B^4 + \kappa_E \sigma_E^4)/(\sigma_B^2 + \sigma_E^2)^2$ is the kurtosis of Y_{ij} for the given i and any j. Thus

$$\mathrm{Var}(s_{i\bullet}^2) = \frac{2(\sigma_B^2 + \sigma_E^2)^2}{N_{i\bullet} - 1} + \frac{\kappa_B \sigma_B^4}{N_{i\bullet}} + \frac{\kappa_E \sigma_E^4}{N_{i\bullet}}. \tag{103}$$

Inverse variance weighting then suggests that we weight $s_{i\bullet}^2$ proportionally to a value between $N_{i\bullet}$ and $N_{i\bullet} - 1$. Weighting proportional to $N_{i\bullet} - 1$ has the advantage of zeroing out rows with $N_{i\bullet} = 1$. This consideration motivates us to take $u_i = 1/N_{i\bullet}$, and similarly $v_j = 1/N_{\bullet j}$.

If U_e is dominated by contributions from e_{ij} then the observations enter symmetrically and there is no reason to not take $w_{ij} = 1$. Even if the e_{ij} do not dominate, the statistic U_e compares more data pairs than the others. It is unlikely to be the information limiting statistic. So $w_{ij} = 1$ is a reasonable default.

If the data are IID then only U_e above is nonzero. This is appropriate as only the sum $\sigma_A^2 + \sigma_B^2 + \sigma_E^2$ can be identified in that case. For data that are nested but not IID, only two of the U-statistics above are nonzero and in that case only one of σ_A^2 and σ_B^2 can be identified separately from σ_E^2.

The U-statistics we use are then

$$\begin{aligned} U_a &= \frac{1}{2} \sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (Y_{ij} - Y_{ij'})^2 \\ U_b &= \frac{1}{2} \sum_{iji'} N_{\bullet j}^{-1} Z_{ij} Z_{i'j} (Y_{ij} - Y_{i'j})^2, \quad \text{and} \\ U_e &= \frac{1}{2} \sum_{iji'j'} Z_{ij} Z_{i'j'} (Y_{ij} - Y_{i'j'})^2. \end{aligned} \tag{104}$$

Because we only sum over i with $N_{i\bullet} > 0$ and j with $N_{\bullet j} > 0$, our sums never include $0/0$.

11.1 Expected U-statistics

Here we find the expected values for our three U-statistics.

Lemma 1. *Under the random effects model* (100)*, the U-statistics in* (104) *satisfy*

$$\begin{pmatrix}\mathbb{E}(U_a)\\ \mathbb{E}(U_b)\\ \mathbb{E}(U_e)\end{pmatrix} = \begin{pmatrix} 0 & N-R & N-R\\ N-C & 0 & N-C\\ N^2-\sum_i N_{i\bullet}^2 & N^2-\sum_j N_{\bullet j}^2 & N^2-N\end{pmatrix}\begin{pmatrix}\sigma_A^2\\ \sigma_B^2\\ \sigma_E^2\end{pmatrix}. \tag{105}$$

Proof. First we note that

$$\begin{aligned}\mathbb{E}((a_i-a_{i'})^2) &= 2\sigma_A^2(1-1_{i=i'})\\ \mathbb{E}((b_j-b_{j'})^2) &= 2\sigma_B^2(1-1_{j=j'}),\quad\text{and}\\ \mathbb{E}((e_{ij}-e_{i'j'})^2) &= 2\sigma_E^2(1-1_{i=i'}1_{j=j'}).\end{aligned}$$

Now $Y_{ij}-Y_{ij'} = b_j-b_{j'}+e_{ij}-e_{ij'}$, and so

$$\begin{aligned}\mathbb{E}(U_a) &= \frac{1}{2}\sum_{ijj'} N_{i\bullet}^{-1}Z_{ij}Z_{ij'}\bigl(2\sigma_B^2(1-1_{j=j'})+2\sigma_E^2(1-1_{i=i}1_{j=j'})\bigr)\\ &= (\sigma_B^2+\sigma_E^2)\sum_{ijj'} N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(1-1_{j=j'})\\ &= (\sigma_B^2+\sigma_E^2)\sum_{ij'} Z_{ij'}(1-1_{j=j'})\\ &= (\sigma_B^2+\sigma_E^2)\sum_i (N_{i\bullet}-1)\\ &= (\sigma_B^2+\sigma_E^2)(N-R).\end{aligned}$$

The same argument give $\mathbb{E}(U_b) = (\sigma_A^2+\sigma_E^2)(N-C)$. □

The matrix in (105) is

$$M \equiv \begin{pmatrix} 0 & N-R & N-R\\ N-C & 0 & N-C\\ N^2-\sum_i N_{i\bullet}^2 & N^2-\sum_j N_{\bullet j}^2 & N^2-N\end{pmatrix}. \tag{106}$$

Our moment based estimates are

$$\begin{pmatrix}\hat\sigma_A^2\\ \hat\sigma_B^2\\ \hat\sigma_E^2\end{pmatrix} = M^{-1}\begin{pmatrix}U_a\\ U_b\\ U_e\end{pmatrix}. \tag{107}$$

They are only well defined when M is nonsingular. The determinant of M is

$$\begin{aligned}&-(N-R)\bigl[(N-C)(N^2-N)-(N-C)(N^2-\sum_i N_{i\bullet}^2)\bigr]\\ &+(N-R)\bigl[(N-C)(N^2-\sum_j N_{\bullet j}^2)\bigr]\\ =&-(N-R)\bigl[(N-C)(\sum_i N_{i\bullet}^2-N)\bigr]+(N-R)\bigl[(N-C)(N^2-\sum_j N_{\bullet j}^2)\bigr]\\ =&\,(N-R)(N-C)[N^2-\sum_i N_{i\bullet}^2-\sum_j N_{\bullet j}^2+N].\end{aligned}$$

The first factor is positive so long as $\max_i N_{i\bullet} > 1$, and the second factor requires $\max_j N_{\bullet j} > 1$. We already knew that we needed these conditions in order to have all three U-statistics depend on the Y_{ij}. It is still of interest to know when the third factor is positive. It is sufficient that no row or column has over half of the data.

12 The variance

From equation (107) we get

$$\mathrm{Var}\begin{pmatrix}\hat\sigma^2_A\\ \hat\sigma^2_B\\ \hat\sigma^2_E\end{pmatrix} = M^{-1}\mathrm{Var}\begin{pmatrix}U_a\\ U_b\\ U_e\end{pmatrix}M^{-1}$$

where M is given at (106). So we need the variances and covariances of the three U statistics.

To find variances, we will work out $\mathbb{E}(U^2)$ for our U-statistics. Those involve

$$\begin{aligned}
&\mathbb{E}((Y_{ij}-Y_{i'j'})^2(Y_{rs}-Y_{r's'})^2)\\
&=\mathbb{E}\big((a_i-a_{i'}+b_j-b_{j'}+e_{ij}-e_{i'j'})^2(a_r-a_{r'}+b_s-b_{s'}+e_{rs}-e_{r's'})^2\big)\\
&=\mathbb{E}\Big[\big((a_i-a_{i'})^2+(b_j-b_{j'})^2+(e_{ij}-e_{i'j'})^2\\
&\quad+2(a_i-a_{i'})(b_j-b_{j'})+2(a_i-a_{i'})(e_{ij}-e_{i'j'})+2(b_j-b_{j'})(e_{ij}-e_{i'j'})\big)\\
&\quad\times\big((a_r-a_{r'})^2+(b_s-b_{s'})^2+(e_{rs}-e_{r's'})^2\\
&\quad+2(a_r-a_{r'})(b_s-b_{s'})+2(a_r-a_{r'})(e_{rs}-e_{r's'})+2(b_s-b_{s'})(e_{rs}-e_{r's'})\big)\Big].
\end{aligned}$$

This expression involves 8 indices and it has 36 terms. Some of those terms simplify due to independence and some vanish due to zero means. To shorten some expressions we use

$$\begin{aligned}
\mathbb{B}_{A,ii',rr'} &\equiv \mathbb{E}((a_i-a_{i'})(a_r-a_{r'}))\\
\mathbb{D}_{A,ii'} &\equiv \mathbb{E}((a_i-a_{i'})^2),\quad\text{and,}\\
\mathbb{Q}_{A,ii',rr'} &\equiv \mathbb{E}((a_i-a_{i'})^2(a_r-a_{r'})^2)
\end{aligned}$$

with mnemonics bilinear, diagonal and quartic. There are similarly defined terms for component B. For the error term we have

$$\begin{aligned}
\mathbb{B}_{E,iji'j',rsr's'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})(e_{rs}-e_{r's'}))\\
\mathbb{D}_{E,ij,i'j'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})^2),\quad\text{and,}\\
\mathbb{Q}_{E,iji'j',rsr's'} &\equiv \mathbb{E}((e_{ij}-e_{i'j'})^2(e_{rs}-e_{r's'})^2).
\end{aligned}$$

The generic contribution $\mathbb{E}((Y_{ij}-Y_{i'j'})^2(Y_{rs}-Y_{r's'})^2)$ to the mean square of a U-statistic equals

$$\begin{aligned}
&\mathbb{Q}_{A,ii',rr'}+\mathbb{Q}_{B,jj',ss'}+\mathbb{Q}_{E,iji'j',rsr's'}\\
&+\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}+\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}\\
&+\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}+\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}\\
&+\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{A,rr'}+\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{B,ss'}\\
&+4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'}+4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}+4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,iji'j',rsr's'}.
\end{aligned}\tag{108}$$

The other 24 terms are zero.

12.1 Variance parts

Here we collect expressions for the quantities appearing in the generic term of our squared U-statistics.

Lemma 2. *In the random effects model* (100)*,*

$$\begin{aligned}
\mathbb{B}_{A,ii',rr'} &= \sigma^2_A\big(1_{i=r}-1_{i=r'}-1_{i'=r}+1_{i'=r'}\big),\\
\mathbb{B}_{B,jj',ss'} &= \sigma^2_B\big(1_{j=s}-1_{j=s'}-1_{j'=s}+1_{j'=s'}\big),\quad\textit{and}\\
\mathbb{B}_{E,iji'j',rsr's'} &= \sigma^2_E\big(1_{ij=rs}-1_{ij=r's'}-1_{i'j'=rs}+1_{i'j'=r's'}\big).
\end{aligned}$$

Proof. The first one follows by expanding and using $\mathbb{E}(a_i a_r) = \sigma_A^2 1_{i=r}$, et cetera. The other two use the same argument. $\square$

Lemma 3. *In the random effects model* (100),

$$\begin{aligned}
\mathbb{D}_{A,ii'} &= 2\sigma_A^2(1 - 1_{i=i'})\\
\mathbb{D}_{B,jj'} &= 2\sigma_B^2(1 - 1_{j=j'})\\
\mathbb{D}_{E,ij,i'j'} &= 2\sigma_E^2(1 - 1_{ij=i'j'}).
\end{aligned}$$

Proof. Take $i = r$ and $i' = r'$ in Lemma 2. $\square$

Lemma 4. *In the random effects model* (100),

$$\begin{aligned}
\mathbb{Q}_{A,ii',rr'} &= 1_{i\neq i'}1_{r\neq r'}\sigma_A^4\Bigl(4 + (\kappa_A+2)(1_{i\in\{r,r'\}} + 1_{i'\in\{r,r'\}}) + 4\times 1_{\{i,i'\}=\{r,r'\}}\Bigr)\\
\mathbb{Q}_{B,jj',ss'} &= 1_{j\neq j'}1_{s\neq s'}\sigma_B^4\Bigl(4 + (\kappa_B+2)(1_{j\in\{s,s'\}} + 1_{j'\in\{s,s'\}}) + 4\times 1_{\{j,j'\}=\{s,s'\}}\Bigr)\\
\mathbb{Q}_{E,iji'j',rsr's'} &= 1_{ij\neq i'j'}1_{rs\neq r's'}\sigma_E^4\Bigl(4 + (\kappa_E+2)(1_{ij\in\{rs,r's'\}} + 1_{i'j'\in\{rs,r's'\}}) + 4\times 1_{\{ij,i'j'\}=\{rs,r's'\}}\Bigr).
\end{aligned}$$

Proof. We prove the first one; the others are similar. This quantity is 0 if $i = i'$ or $r = r'$. When $i \neq i'$ and $r \neq r'$, there are 3 cases to consider: $|\{i,i'\}\cap\{r,r'\}| = 0$, $|\{i,i'\}\cap\{r,r'\}| = 1$ and $|\{i,i'\}\cap\{r,r'\}| = 2$. The kurtosis is defined via $\kappa_A = \mathbb{E}(a^4)/\sigma_A^4 - 3$, so $\mathbb{E}(a^4) = (\kappa_A + 3)\sigma_A^4$.

For no overlap, we find

$$\mathbb{E}((a_1 - a_2)^2(a_3 - a_4)^2) = \mathbb{E}((a_1 - a_2)^2)^2 = 4\sigma_A^4.$$

For a single overlap,

$$\begin{aligned}
\mathbb{E}((a_1 - a_2)^2(a_1 - a_3)^2) &= \mathbb{E}((a_1^2 - 2a_1a_2 + a_2^2)(a_1^2 - 2a_1a_3 + a_3^2))\\
&= \mathbb{E}(a_1^4) + 3\sigma_A^4 = \sigma_A^4(\kappa_A + 6).
\end{aligned}$$

For a double overlap,

$$\begin{aligned}
\mathbb{E}((a_1 - a_2)^4) &= \mathbb{E}(a_1^4 - 4a_1a_2^3 + 6a_1^2a_2^2 - 4a_1^3a_2 + a_2^4)\\
&= 2\mathbb{E}(a_1^4) + 6\sigma_A^4 = \sigma_A^4(2\kappa_A + 12).
\end{aligned}$$

As a result,

$$\mathbb{E}((a_i - a_{i'})^2(a_r - a_{r'})^2) = \begin{cases} 4\sigma_A^4, & |\{i,i'\}\cap\{r,r'\}| = 0,\\ \sigma_A^4(\kappa_A + 6), & |\{i,i'\}\cap\{r,r'\}| = 1,\\ \sigma_A^4(2\kappa_A + 12), & |\{i,i'\}\cap\{r,r'\}| = 2,\end{cases}$$

and so $\mathbb{E}((a_i - a_{i'})^2(a_r - a_{r'})^2)$ equals

$$1_{i\neq i'}1_{r\neq r'}\sigma_A^4\Bigl(4 + (\kappa_A+2)(1_{i\in\{r,r'\}} + 1_{i'\in\{r,r'\}}) + 4\times 1_{\{i,i'\}=\{r,r'\}}\Bigr). \quad \square$$

12.2 Variance of U_a

We will work out $\mathbb{E}(U_a^2)$ and then subtract $\mathbb{E}(U_a)^2$. First we write

$$U_a^2 = \frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(Y_{ij} - Y_{ij'})^2(Y_{rs} - Y_{rs'})^2.$$

For $\mathbb{E}(U_a^2)$ we use the special case $i = i'$ and $r = r'$ of (108),

$$\mathbb{E}(U_a^2) = \frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\Bigl[$$

$$
\begin{aligned}
&\mathbb{Q}_{A,ii,rr} + \mathbb{Q}_{B,jj',ss'} + \mathbb{Q}_{E,ijij',rsrs'} \\
&+ \mathbb{D}_{A,ii}\mathbb{D}_{B,ss'} + \mathbb{D}_{A,ii}\mathbb{D}_{E,rs,rs'} \\
&+ \mathbb{D}_{B,jj'}\mathbb{D}_{A,rr} + \mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,rs'} \\
&+ \mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr} + \mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'} \\
&+ 4\mathbb{B}_{A,ii,rr}\mathbb{B}_{B,jj',ss'} + 4\mathbb{B}_{A,ii,rr}\mathbb{B}_{E,ijij',rsrs'} + 4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsrs'}\Big] \\
= &\frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\Big[\underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 1}} + \underbrace{\mathbb{Q}_{E,ijij',rsrs'}}_{\text{Term 2}} \\
&+ \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,rs'}}_{\text{Term 3}} + \underbrace{\mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}}_{\text{Term 4}} + \underbrace{4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsrs'}}_{\text{Term 5}}\Big]
\end{aligned}
$$

after eliminating terms that are always 0. We handle these five sums in the next subsubsections.

12.2.1 U_a^2 term 1

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{Q}_{B,jj',ss'} \\
&= \frac{\sigma_B^4}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j\neq j'}1_{s\neq s'} \\
&\quad\Big(4 + (\kappa_B+2)(1_{j\in\{s,s'\}} + 1_{j'\in\{s,s'\}}) + 4\times 1_{\{j,j'\}=\{s,s'\}}\Big) \\
&= \frac{\sigma_B^4}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(1-1_{j=j'})(1-1_{s=s'}) \\
&\quad\Big(\underbrace{4}_{1.1} + \underbrace{(\kappa_B+2)(1_{j\in\{s,s'\}} + 1_{j'\in\{s,s'\}})}_{1.2\text{ and }1.3} + \underbrace{4\times 1_{\{j,j'\}=\{s,s'\}}}_{1.4}\Big).
\end{aligned}
$$

Term 1 is now a sum of four terms, 1.1 through 1.4. Term 1.1 is σ_B^4 times

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}4(1-1_{j=j'}-1_{s=s'}+1_{j=j'}1_{s=s'}) \\
&= \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'} \\
&\quad - \sum_{ij}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{rs}Z_{rs'} \\
&\quad - \sum_{ijj'}\sum_{rs} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs} \\
&\quad + \sum_{ij}\sum_{rs} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{rs} \\
&= \sum_{ir}(N_{i\bullet}N_{r\bullet} - N_{r\bullet} - N_{i\bullet} + 1) \\
&= (N-R)^2.
\end{aligned}
$$

Term 1.2 is $\sigma_B^4(\kappa_B+2)/4$ times

$$
\begin{aligned}
&\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(1-1_{j=j'}-1_{s=s'}+1_{j=j'}1_{s=s'})1_{j\in\{s,s'\}} \\
&= \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}(1_{j=s}+1_{j=s'}-1_{j=s}1_{j=s'})
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{ij}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{rs} Z_{rs'} (1_{j=s} + 1_{j=s'} - 1_{j=s} 1_{j=s'}) \\
&- \sum_{ijj'}\sum_{rs} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} 1_{j=s} \\
&+ \sum_{ij}\sum_{rs} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{rs} 1_{j=s} \\
= {}& 2 \sum_{ir} (ZZ^\mathsf{T})_{ir} - \sum_{ir} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} \\
&- 2 \sum_{ir} N_{i\bullet}^{-1} (ZZ^\mathsf{T})_{ir} + \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} \\
&- \sum_{ir} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} + \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} \\
= {}& 2 \sum_{ir} (ZZ^\mathsf{T})_{ir} (1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1}).
\end{aligned}
$$

The expression $\sum_{ir}(ZZ^\mathsf{T})_{ir}$ simplifies to $\sum_j N_{\bullet j}^2$, changing it from a 'row quantity' to a 'column quantity'. But the other parts of this expression are equivalent to sums of terms such as $N_{i\bullet}^{-1} Z_{ij} N_{\bullet j}$ making the column version less convenient to work with. Term 1.3 is the same as term 1.2 by symmetry of indices.

Term 1.4 is σ_B^4 times

$$
\begin{aligned}
&\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} (1 - 1_{j=j'})(1 - 1_{s=s'}) 1_{\{j,j'\}=\{s,s'\}} \\
= {}& \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{j\neq j'} 1_{s\neq s'} 1_{\{j,j'\}=\{s,s'\}} \\
= {}& 2 \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{j\neq j'} 1_{s\neq s'} 1_{j=s} 1_{j'=s'} \\
= {}& 2 \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{rj'} 1_{j\neq j'} \\
= {}& 2 \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{rj'} - 2 \sum_{ij}\sum_{r} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{rj} \\
= {}& 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir}^2 - 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir}.
\end{aligned}
$$

Summing terms 1.1 through 1.4 yields

$$
\begin{aligned}
&\sigma_B^4 \Big((N-R)^2 + (\kappa_B + 2) \sum_{ir} (ZZ^\mathsf{T})_{ir} (1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1}) \\
&+ 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} ((ZZ^\mathsf{T})_{ir} - 1) \Big).
\end{aligned}
$$

12.2.2 U_a^2 term 2

$$
\begin{aligned}
&\frac{1}{4} \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} \mathbb{Q}_{E,ijij',rsrs'} \\
= {}& \frac{\sigma_E^4}{4} \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{ij\neq ij'} 1_{rs\neq rs'} \\
&\times \Big(4 + (\kappa_E + 2)(1_{ij\in\{rs,rs'\}} + 1_{ij'\in\{rs,rs'\}}) + 4 \times 1_{\{ij,ij'\}=\{rs,rs'\}} \Big) \\
= {}& \frac{\sigma_E^4}{4} \sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{j\neq j'} 1_{s\neq s'}
\end{aligned}
$$

$$\times \Big(\underbrace{4}_{2.1} + \underbrace{(\kappa_E + 2) 1_{i=r} (1_{j \in \{s,s'\}} + 1_{j' \in \{s,s'\}})}_{2.2 \text{ and } 2.3} + \underbrace{4 1_{i=r} 1_{\{j,j'\}=\{s,s'\}}}_{2.4} \Big).$$

Term 2.1 is σ_E^4 times

$$\sum_{ijj'} \sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{j \neq j'} 1_{s \neq s'} = (N - R)^2$$

by the same process that evaluated term 1.1.

Term 2.2 is $\sigma_E^4(\kappa_E + 2)/4$ times

$$\begin{aligned}
&\sum_{ijj'} \sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{j \neq j'} 1_{s \neq s'} 1_{i=r} 1_{j \in \{s,s'\}} \\
= &\sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j \in \{s,s'\}} \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} 1_{j \in \{s,s'\}} \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{s=s'} 1_{j \in \{s,s'\}} \\
&+ \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} 1_{s=s'} 1_{j \in \{s,s'\}}
\end{aligned}$$

which reduces to

$$\begin{aligned}
&\sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j \in \{s,s'\}} \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} 1_{j \in \{s,s'\}} \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{s=s'} 1_{j=s} \\
&+ \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} 1_{j=s} 1_{j=s'} \\
= &\sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} (1_{j=s} + 1_{j=s'} - 1_{j=s} 1_{j=s'}) \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} (1_{j=s} + 1_{j=s'} - 1_{j=s} 1_{j=s'}) \\
&- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{s=s'} 1_{j=s} \\
&+ \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=j'} 1_{j=s} 1_{j=s'} \\
= &\,2 \sum_{ijj'} \sum_{s} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} - \sum_{ijj'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} \\
&- 2 \sum_{ij} \sum_{s} N_{i\bullet}^{-2} Z_{ij} Z_{is} + \sum_{ij} N_{i\bullet}^{-2} Z_{ij} \\
&- \sum_{ijj'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} + \sum_{ij} N_{i\bullet}^{-2} Z_{ij} \\
= &\,2 \sum_{i} N_{i\bullet} - R - 2R + \sum_{i} N_{i\bullet}^{-1} - R + \sum_{i} N_{i\bullet}^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= 2N - 4R + 2\sum_i N_{i\bullet}^{-1} \\
&= 2\sum_i N_{i\bullet}(1 - N_{i\bullet}^{-1})^2.
\end{aligned}
$$

The last expression resembles the diagonal part of term 1.2. Term 2.3 is the same is the same as term 2.2.

Term 2.4 is σ_E^4 times

$$
\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} 1_{i=r} 1_{j\neq j} 1_{s\neq s'} 1_{\{j,j'\}=\{s,s'\}}
$$

This is the same sum as the coefficient in term 1.4 has except that it has the additional constraint $i = r$. Imposing $i = r$ on that quantity yields

$$
2\sum_i N_{i\bullet}^{-2}(ZZ^{\mathsf{T}})_{ii}^2 - 2\sum_i N_{i\bullet}^{-2}(ZZ^{\mathsf{T}})_{ii} = 2\sum_i (1 - N_{i\bullet}^{-1}).
$$

Term 2 is thus

$$
\sigma_E^4\Bigl((N-R)^2 + (\kappa_E + 2)\sum_i N_{i\bullet}(1 - N_{i\bullet}^{-1})^2 + 2\sum_i (1 - N_{i\bullet}^{-1})\Bigr).
$$

12.2.3 U_a^2 terms 3 and 4

These terms are equal by symmetry. We evaluate term 3.

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} \mathbb{D}_{B,jj'} \mathbb{D}_{E,rs,rs'} \\
&= \frac{1}{4}\Bigl(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} \mathbb{D}_{B,jj'}\Bigr)\Bigl(\sum_{rss'} N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \mathbb{D}_{E,rs,rs'}\Bigr).
\end{aligned}
$$

Now

$$
\begin{aligned}
\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} \mathbb{D}_{B,jj'} &= 2\sigma_B^2 \sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'}(1 - 1_{j=j'}) \\
&= 2\sigma_B^2 \sum_i (N_{i\bullet} - 1) = 2\sigma_B^2(N - R)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{rss'} N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \mathbb{D}_{E,rs,rs'} &= 2\sigma_E^2 \sum_{rss'} N_{i\bullet}^{-1} Z_{rs} Z_{rs'}(1 - 1_{s=s'}) \\
&= 2\sigma_E^2(N - R)
\end{aligned}
$$

by the same steps. Therefore term 3 of $\mathbb{E}(U_a^2)$ equals $\sigma_B^2\sigma_E^2(N-R)^2$ and the sum of terms 3 and 4 is $2\sigma_B^2\sigma_E^2(N-R)^2$.

12.2.4 U_a^2 term 5

The term equals

$$
\begin{aligned}
&\sum_{ijj'}\sum_{rss'} N_{i\bullet}^{-1} N_{r\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{rs'} \mathbb{B}_{B,jj',ss'} \mathbb{B}_{E,ijij',rsrs'} \\
&= \sum_{ijj'}\sum_{ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} \mathbb{B}_{B,jj',ss'} \sum_r N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \mathbb{B}_{E,ijij',rsrs'}.
\end{aligned}
$$

Now

$$\sum_r N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \mathbb{B}_{E,ijij',rsrs'} = \sigma_E^2 \sum_r N_{r\bullet}^{-1} Z_{rs} Z_{rs'} \big(1_{ij=rs} - 1_{ij=rs'} - 1_{ij'=rs} + 1_{ij'=rs'}\big)$$
$$= \sigma_E^2 N_{i\bullet}^{-1} Z_{is} Z_{is'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big).$$

Term 5 is then

$$\sigma_E^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \mathbb{B}_{B,jj',ss'}$$
$$= \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)^2$$
$$= \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$- \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$- \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j'=s} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$+ \sigma_E^2 \sigma_B^2 \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j'=s'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big),$$

which we call terms 5.1, 5.2, 5.3 and 5.4. Next we find the coefficients of $\sigma_B^2\sigma_E^2$ in these four terms.

For term 5.1, we get

$$\sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$= \sum_{ijj'} \sum_{s'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is'} \big(1 - 1_{j=s'} - 1_{j=j'} + 1_{j'=s'}\big)$$
$$= \sum_i (N_{i\bullet} - 1)$$
$$= N - R.$$

For term 5.2, we get

$$- \sum_{ijj'} \sum_{ss'} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} Z_{is'} 1_{j=s'} \big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$= - \sum_{ijj'} \sum_{s} N_{i\bullet}^{-2} Z_{ij} Z_{ij'} Z_{is} \big(1_{j=s} - 1 - 1_{j'=s} + 1_{j=j'}\big)$$
$$= N - R$$

as well. Terms 5.3 and 5.4 are also $N_{i\bullet} - 1$, by the steps used for terms 5.2 and 5.1 respectively. As a result term 5 equals $4\sigma_B^2\sigma_E^2(N-R)$.

12.3 Combination

Combining the results of the previous sections, we have

$$\mathbb{E}(U_a^2) = \sigma_B^4 \Big((N-R)^2 + (\kappa_B + 2) \sum_{ir} (ZZ^\mathsf{T})_{ir} (1 - N_{i\bullet}^{-1})(1 - N_{r\bullet}^{-1})$$
$$+ 2 \sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^\mathsf{T})_{ir} ((ZZ^\mathsf{T})_{ir} - 1) \Big)$$
$$+ 2\sigma_B^2 \sigma_E^2 (N-R)^2 + 4\sigma_B^2 \sigma_E^2 (N-R)$$

$$+\sigma_E^4\Big((N-R)^2+(\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2\sum_i(1-N_{i\bullet}^{-1})\Big).$$

Subtracting $\mathbb{E}(U_a)^2=(N-R)^2(\sigma_B^2+\sigma_E^2)^2$ we find

$$\begin{aligned}\mathrm{Var}(U_a)=\sigma_B^4\Big((\kappa_B+2)\sum_{ir}(ZZ^\mathsf{T})_{ir}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1})+2\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir}((ZZ^\mathsf{T})_{ir}-1)\Big)\\+4\sigma_B^2\sigma_E^2(N-R)+\sigma_E^4\Big((\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2\sum_i(1-N_{i\bullet}^{-1})\Big).\end{aligned}\tag{109}$$

12.4 Checks

We can check some special cases of this formula.

12.4.1 Rows nested in columns

If for instance rows are nested within columns, then $N=R$, and all $N_{i\bullet}=N_{r\bullet}=1$ and in this case $U_a=0$. The above formula gives $\mathrm{Var}(U_a)=0$ for this case.

12.4.2 Columns nested in rows

If columns are nested in rows, then $(ZZ^\mathsf{T})_{ir}=1_{i=r}N_{i\bullet}$ and equation (109) yields

$$\begin{aligned}\mathrm{Var}(U_a)&=\sigma_B^4\Big((\kappa_B+2)\sum_{ir}N_{i\bullet}1_{i=r}(1-N_{i\bullet}^{-1})(1-N_{r\bullet}^{-1})+2\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}1_{i=r}N_{i\bullet}(N_{i\bullet}-1)\Big)\\&\quad+4\sigma_B^2\sigma_E^2(N-R)+\sigma_E^4\Big((\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2\sum_i(1-N_{i\bullet}^{-1})\Big)\\&=\Big(\sigma_B^4(\kappa_B+2)+\sigma_E^4(\kappa_E+2)\Big)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2(\sigma_B^4+\sigma_E^4)\sum_i(1-N_{i\bullet}^{-1})+4\sigma_B^2\sigma_E^2(N-R)\\&=(\kappa_B\sigma_B^4+\kappa_E\sigma_E^4)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+(\sigma_B^4+\sigma_E^4)\sum_i\Big(2N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2(1-N_{i\bullet}^{-1})\Big)+4\sigma_B^2\sigma_E^2(N-R)\\&=(\kappa_B\sigma_B^4+\kappa_E\sigma_E^4)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2(\sigma_B^4+\sigma_E^4)\sum_i(N_{i\bullet}-1)+4\sigma_B^2\sigma_E^2(N-R)\\&=(\kappa_B\sigma_B^4+\kappa_E\sigma_E^4)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2+2(N-R)(\sigma_B^2+\sigma_E^2)^2.\end{aligned}\tag{110}$$

When columns are nested in rows, then $U_a=\sum_i(N_{i\bullet}-1)s_{i\bullet}^2$ and because the rows are then independent, U_a has variance

$$(\sigma_B^2+\sigma_E^2)^2\sum_i(N_{i\bullet}-1)^2\Big(\frac{2}{N_{i\bullet}-1}+\frac{\kappa(b_1+e_{11})}{N_{i\bullet}}\Big).$$

The kurtosis of b_j+e_{ij} is

$$\kappa_{B+E}=\kappa_B\Big(\frac{\sigma_B^2}{\sigma_B^2+\sigma_E^2}\Big)^2+\kappa_E\Big(\frac{\sigma_E^2}{\sigma_B^2+\sigma_E^2}\Big)^2.$$

Therefore for columns nested in rows

$$\begin{aligned}\mathrm{Var}(U_a)&=(\sigma_B^2+\sigma_E^2)^2\sum_i\Big(2(N_{i\bullet}-1)+\frac{(N_{i\bullet}-1)^2}{N_{i\bullet}}\Big(\kappa_B\Big(\frac{\sigma_B^2}{\sigma_B^2+\sigma_E^2}\Big)^2+\kappa_E\Big(\frac{\sigma_E^2}{\sigma_B^2+\sigma_E^2}\Big)^2\Big)\Big)\\&=2(\sigma_B^2+\sigma_E^2)^2\sum_i(N_{i\bullet}-1)+(\kappa_B\sigma_B^4+\kappa_E\sigma_E^4)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2\\&=2(N-R)(\sigma_B^2+\sigma_E^2)^2+(\kappa_B\sigma_B^4+\kappa_E\sigma_E^4)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2,\end{aligned}$$

matching the expression (110) that comes from equation (109) for $\mathrm{Var}(U_a)$.

12.4.3 $\sigma_B^2 = 0$

If $\sigma_B^2 = 0$ then $\mathrm{Var}(U_a)$ should be the same as it is for columns nested in rows. In this case equation (109) reduces to

$$\begin{aligned}
\mathrm{Var}(U_a) &= \sigma_E^4\Big((\kappa_E+2)\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2 + 2\sum_i(1-N_{i\bullet}^{-1})\Big)\\
&= \sigma_E^4\Big(\kappa_E\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2 + 2\sum_i\Big(N_{i\bullet}(1-N_{i\bullet}^{-1})^2+(1-N_{i\bullet}^{-1})\Big)\Big)\\
&= \sigma_E^4\Big(\kappa_E\sum_i N_{i\bullet}(1-N_{i\bullet}^{-1})^2 + 2(N-R)\Big).
\end{aligned}$$

If instead we first take the columns nested in rows special case from equation (110) and then substitute $\sigma_B^2 = 0$, we get the same expression.

12.4.4 $\sigma_E^2 = 0$ and $\kappa_B = -2$

In this special case we take $\sigma_E^2 = 0$ and take $b_j \sim U(\pm 1)$. Then $\sigma_B^2 = 1$ and $\kappa_B = -2$. Then

$$\mathrm{Var}(U_a) = 2\sigma_B^4\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir}((ZZ^\mathsf{T})_{ir}-1) = 2\sum_{ir}N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir}((ZZ^\mathsf{T})_{ir}-1).$$

In this case

$$\begin{aligned}
U_a &= \frac{1}{2}\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(b_j-b_{j'})^2\\
&= \frac{1}{2}\sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(b_j^2-2b_jb_{j'}+b_{j'}^2)\\
&= \sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(1-b_jb_{j'})\\
&= \sum_i N_{i\bullet} - \sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}b_jb_{j'}
\end{aligned}$$

and so $\mathrm{Var}(U_a) = \mathrm{Var}(\widetilde{U}_a)$ where $\widetilde{U}_a = \sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}b_jb_{j'}$. We easily find that

$$\mathbb{E}(\widetilde{U}_a) = \sum_{ijj'}N_{i\bullet}^{-1}Z_{ij}Z_{ij'}1_{j=j'} = \sum_i 1 = R.$$

To get the variance of $\widetilde{U}_a$ we need

$$\mathbb{E}(b_jb_{j'}b_sb_{s'}) = 1_{j=j'}1_{s=s'} + 1_{j=s}1_{j'=s'} + 1_{j=s'}1_{j'=s} - 2\times 1_{j=s'}1_{j'=s'}1_{s=s'}.$$

Now

$$\begin{aligned}
\mathbb{E}(\widetilde{U}_a^2) &= \sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\mathbb{E}(b_jb_{j'}b_sb_{s'})\\
&= \sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j=j'}1_{s=s'}\\
&\quad+\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j=s}1_{j'=s'}\\
&\quad+\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j=s'}1_{j'=s}\\
&\quad-2\sum_{ijj'}\sum_{rss'}N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{j=s'}1_{j'=s'}1_{s=s'}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{ij}\sum_{rs} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{rs} + \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rj}Z_{rj'} \\
&\quad + \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{ij'}Z_{rj'}Z_{rj} - 2\sum_{ij}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}Z_{ij}Z_{rj} \\
&= \sum_{i}\sum_{r} 1 + 2\sum_{i}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir}^2 - 2\sum_{i}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir} \\
&= R^2 + 2\sum_{i}\sum_{r} N_{i\bullet}^{-1}N_{r\bullet}^{-1}(ZZ^\mathsf{T})_{ir}((ZZ^\mathsf{T})_{ir} - 1).
\end{aligned}
$$

In this case we get

$$
\mathrm{Var}(\widetilde{U}_a) = R^2 + 2\sum_{i}\sum_{r}(ZZ^\mathsf{T})_{ir}((ZZ^\mathsf{T})_{ir} - 1) - R^2
$$

matching the result from (109).

12.4.5 Crossed design

In a crossed design $N_{i\bullet} = C$ for all i and $(ZZ^\mathsf{T})_{ir} = C$ for all i and r. Here the variance is

$$
\begin{aligned}
\mathrm{Var}(U_a) &= \sigma_B^4\Big((\kappa_B + 2)\sum_{ir} C(1 - C^{-1})^2 + 2\sum_{ir} C^{-1}(C - 1)\Big) \\
&\quad + \sigma_E^4\Big((\kappa_E + 2)\sum_{i} C(1 - C^{-1})^2 + 2\sum_{i}(1 - C^{-1})\Big) + 4\sigma_B^2\sigma_E^2(N - R) \\
&= \sigma_B^4\Big((\kappa_B + 2)C(1 - C^{-1})^2 + 2(1 - C^{-1})\Big)R^2 \\
&\quad + \sigma_E^4\Big((\kappa_E + 2)C(1 - C^{-1})^2 + 2(1 - C^{-1})\Big)R + 4\sigma_B^2\sigma_E^2(R - 1)C.
\end{aligned}
\tag{111}
$$

12.5 Variance of U_b

This case is exactly symmetric to the one above with $\mathrm{Var}(U_a)$ given by (109). Therefore

$$
\begin{aligned}
\mathrm{Var}(U_b) = \sigma_B^4\Big((\kappa_A + 2)\sum_{js}(Z^\mathsf{T}Z)_{js}(1 - N_{\bullet j}^{-1})(1 - N_{\bullet s}^{-1}) + 2\sum_{js} N_{\bullet j}^{-1}N_{\bullet s}^{-1}(Z^\mathsf{T}Z)_{js}((Z^\mathsf{T}Z)_{js} - 1)\Big) \\
+ \sigma_E^4\Big((\kappa_E + 2)\sum_{j} N_{\bullet j}(1 - N_{\bullet j}^{-1})^2 + 2\sum_{j}(1 - N_{\bullet j}^{-1})\Big).
\end{aligned}
\tag{112}
$$

12.6 Variance of U_e

As before, we find $\mathbb{E}(U_e^2)$ and then subtract $\mathbb{E}(U_e)^2$. Now

$$
U_e^2 = \frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(Y_{ij} - Y_{i'j'})^2(Y_{rs} - Y_{r's'})^2.
$$

From (108),

$$
\begin{aligned}
\mathbb{E}(U_e^2) = \frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\Big[&\underbrace{\mathbb{Q}_{A,ii',rr'}}_{\text{Term 1}} + \underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 2}} + \underbrace{\mathbb{Q}_{E,iji'j',rsr's'}}_{\text{Term 3}} \\
+ \underbrace{\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}}_{\text{Term 4}} + \underbrace{\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}}_{\text{Term 5}} &+ \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}}_{\text{Term 6}} + \underbrace{\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}}_{\text{Term 7}} \\
+ \underbrace{\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{A,rr'}}_{\text{Term 8}} + \underbrace{\mathbb{D}_{E,ij,i'j'}\mathbb{D}_{B,ss'}}_{\text{Term 9}}&
\end{aligned}
$$

$$+\underbrace{4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'}}_{\text{Term 10}}+\underbrace{4\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}}_{\text{Term 11}}+\underbrace{4\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,iji'j',rsr's'}}_{\text{Term 12}}\Big].$$

We handle the twelve sums in the next subsections.

12.6.1 U_e^2 Term 1

As before, we split term 1 into four parts.

$$\begin{aligned}
&\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{Q}_{A,ii',rr'}\\
&=\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{i\neq i'}1_{r\neq r'}\sigma_A^4\Big(4+(\kappa_A+2)(1_{i\in\{r,r'\}}+1_{i'\in\{r,r'\}})+4\times 1_{\{i,i'\}=\{r,r'\}}\Big)\\
&=\frac{\sigma_A^4}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{i=i'})(1-1_{r=r'})\\
&\quad\Big(\underbrace{4}_{1.1}+\underbrace{(\kappa_A+2)(1_{i\in\{r,r'\}}+1_{i'\in\{r,r'\}})}_{1.2\text{ and }1.3}+\underbrace{4\times 1_{\{i,i'\}=\{r,r'\}}}_{1.4}\Big).
\end{aligned}$$

For term 1.1, we have σ_A^4 times

$$\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{i=i'}-1_{r=r'}+1_{i=i'}1_{r=r'})\\
&=\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}-\sum_{ijj'}\sum_{rr'ss'} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}-\sum_{ii'jj'}\sum_{rss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{rs'}+\sum_{ijj'}\sum_{rss'} Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}\\
&=N^4-N^2\sum_i N_{i\bullet}^2-N^2\sum_r N_{r\bullet}^2+\Big(\sum_i N_{i\bullet}^2\Big)\Big(\sum_r N_{i\bullet}^2\Big)\\
&=\Big(N^2-\sum_i N_{i\bullet}^2\Big)^2.
\end{aligned}$$

Term 1.2 is $\sigma_A^4(\kappa_A+2)/4$ times

$$\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{i=i'}-1_{r=r'}+1_{i=i'}1_{r=r'})1_{i\in\{r,r'\}}\\
&=\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1_{i=r}+1_{i=r'}-1_{i=r}1_{i=r'})\\
&\quad-\sum_{ijj'}\sum_{rr'ss'} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}(1_{i=r}+1_{i=r'}-1_{i=r}1_{i=r'})\\
&\quad-\sum_{ii'jj'}\sum_{rss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{rs'}1_{i=r}\\
&\quad+\sum_{ijj'}\sum_{rss'} Z_{ij}Z_{ij'}Z_{rs}Z_{rs'}1_{i=r}\\
&=\sum_{ii'jj'}\sum_{r'ss'} Z_{ij}Z_{i'j'}Z_{is}Z_{r's'}+\sum_{ii'jj'}\sum_{rss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{is'}-\sum_{ii'jj'}\sum_{ss'} Z_{ij}Z_{i'j'}Z_{is}Z_{is'}\\
&\quad-\sum_{ijj'}\sum_{r'ss'} Z_{ij}Z_{ij'}Z_{is}Z_{r's'}-\sum_{ijj'}\sum_{rss'} Z_{ij}Z_{ij'}Z_{rs}Z_{is'}+\sum_{ijj'}\sum_{ss'} Z_{ij}Z_{ij'}Z_{is}Z_{is'}\\
&\quad-\sum_{ii'jj'}\sum_{ss'} Z_{ij}Z_{i'j'}Z_{is}Z_{is'}+\sum_{ijj'}\sum_{ss'} Z_{ij}Z_{ij'}Z_{is}Z_{is'}\\
&=N^2\sum_i N_{i\bullet}^2+N^2\sum_i N_{i\bullet}^2-N\sum_i N_{i\bullet}^3
\end{aligned}$$

$$
\begin{aligned}
&- N\sum_i N_{i\bullet}^3 - N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4 \\
&- N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4 \\
= {}& 2N^2\sum_i N_{i\bullet}^2 - 4N\sum_i N_{i\bullet}^3 + 2\sum_i N_{i\bullet}^4.
\end{aligned}
$$

By symmetry of indices, term 1.3 is the same as term 1.2.

For term 1.4, we have σ_A^4 times

$$
\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{i\neq i'}1_{r\neq r'}1_{\{i,i'\}=\{r,r'\}} \\
&= 2\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{i\neq i'}1_{r\neq r'}1_{i=r}1_{i'=r'} \\
&= 2\sum_{ii'jj'}\sum_{ss'} Z_{ij}Z_{i'j'}Z_{is}Z_{i's'}1_{i\neq i'} \\
&= 2\sum_{ii'} N_{i\bullet}N_{i'\bullet}N_{i\bullet}N_{i'\bullet}(1-1_{i=i'}) \\
&= 2\Big(\sum_i N_{i\bullet}^2\Big)^2 - 2\sum_i N_{i\bullet}^4.
\end{aligned}
$$

Summing terms 1.1 to 1.4 gives

$$
\begin{aligned}
&\sigma_A^4\Big(N^4 - 2N^2\sum_i N_{i\bullet}^2 + 3\Big(\sum_i N_{i\bullet}^2\Big)^2 - 2\sum_i N_{i\bullet}^4\Big) \\
&+ \sigma_A^4(\kappa_A+2)\Big(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Big).
\end{aligned}
$$

12.6.2 U_e^2 Term 2

We can use the symmetry of the roles of A and B and their indices. Therefore, term 2 is equal to

$$
\begin{aligned}
&\sigma_B^4\Big(N^4 - 2N^2\sum_j N_{\bullet j}^2 + 3\Big(\sum_j N_{\bullet j}^2\Big)^2 - 2\sum_j N_{\bullet j}^4\Big) \\
&+ \sigma_B^4(\kappa_B+2)\Big(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Big).
\end{aligned}
$$

12.6.3 U_e^2 Term 3

As before, we split term 3 into four parts.

$$
\begin{aligned}
&\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{Q}_{E,iji'j',rsr's'} \\
&= \frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{ij\neq i'j'}1_{rs\neq r's'}\sigma_E^4\Big(4 + (\kappa_E+2)(1_{ij\in\{rs,r's'\}} + 1_{i'j'\in\{rs,r's'\}}) + 4\times 1_{\{ij,i'j'\}=\{rs,r's'\}}\Big) \\
&= \frac{\sigma_E^4}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1-1_{ij=i'j'})(1-1_{rs=r's'}) \\
&\quad\Big(\underbrace{4}_{3.1} + \underbrace{(\kappa_E+2)(1_{ij\in\{rs,r's'\}} + 1_{i'j'\in\{rs,r's'\}})}_{3.2\text{ and }3.3} + \underbrace{4\times 1_{\{ij,i'j'\}=\{rs,r's'\}}}_{3.4}\Big).
\end{aligned}
$$

For term 3.1, we have σ_E^4 times

$$
\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1 - 1_{ij=i'j'} - 1_{rs=r's'} + 1_{ij=i'j'}1_{rs=r's'}) \\
&= \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'} - \sum_{ij}\sum_{rr'ss'} Z_{ij}Z_{rs}Z_{r's'} - \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs} + \sum_{ij}\sum_{rs} Z_{ij}Z_{rs} \\
&= N^4 - N^3 - N^3 + N^2 \\
&= N^2(N-1)^2.
\end{aligned}
$$

Term 3.2 is $\sigma_E^4(\kappa_E + 2)/4$ times

$$
\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1 - 1_{ij=i'j'} - 1_{rs=r's'} + 1_{ij=i'j'}1_{rs=r's'})1_{ij\in\{rs,r's'\}} \\
&= \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1_{ij=rs} + 1_{ij=r's'} - 1_{ij=rs}1_{ij=r's'}) \\
&\quad - \sum_{ij}\sum_{rr'ss'} Z_{ij}Z_{rs}Z_{r's'}(1_{ij=rs} + 1_{ij=r's'} - 1_{ij=rs}1_{ij=r's'}) \\
&\quad - \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}1_{ij=rs} \\
&\quad + \sum_{ij}\sum_{rs} Z_{ij}Z_{rs}1_{ij=rs} \\
&= \sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'} + \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs} - \sum_{ii'jj'} Z_{ij}Z_{i'j'} \\
&\quad - \sum_{ij}\sum_{r's'} Z_{ij}Z_{r's'} - \sum_{ij}\sum_{rs} Z_{ij}Z_{rs} + \sum_{ij} Z_{ij} \\
&\quad - \sum_{ii'jj'} Z_{ij}Z_{i'j'} + \sum_{ij} Z_{ij} \\
&= N^3 + N^3 - N^2 \\
&\quad - N^2 - N^2 + N \\
&\quad - N^2 + N \\
&= 2N^3 - 4N^2 + 2N \\
&= 2N(N-1)^2.
\end{aligned}
$$

By symmetry of indices, term 3.3 is the same as term 3.2.

For term 3.4, we have σ_E^4 times

$$
\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{ij\neq i'j'}1_{rs\neq r's'}1_{\{ij,i'j'\}=\{rs,r's'\}} \\
&= 2\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}1_{ij\neq i'j'}1_{rs\neq r's'}1_{ij=rs}1_{i'j'=r's'} \\
&= 2\sum_{ii'jj'} Z_{ij}Z_{i'j'}1_{ij\neq i'j'} \\
&= 2N(N-1).
\end{aligned}
$$

Summing terms 3.1 to 3.4, we get

$$
\sigma_E^4 N(N-1)[N(N-1)+2] + \sigma_E^4(\kappa_E+2)N(N-1)^2.
$$

12.6.4 U_e^2 Term 4

$$\begin{aligned}&\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{D}_{A,ii'}\mathbb{D}_{B,ss'}\\&=\frac{1}{4}\Big(\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'}\Big)\Big(\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{B,ss'}\Big).\end{aligned}$$

The first factor is

$$\begin{aligned}\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'} &= 2\sigma_A^2\sum_{ii'jj'} Z_{ij}Z_{i'j'}(1-1_{i=i'})\\&=2\sigma_A^2(N^2-\sum_{ijj'} Z_{ij}Z_{ij'})\\&=2\sigma_A^2(N^2-\sum_i N_{i\bullet}^2).\end{aligned}$$

By the same argument, the second factor is

$$\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{B,ss'} = 2\sigma_B^2(N^2-\sum_s N_{\bullet s}^2),$$

and so term 4 is

$$\sigma_A^2\sigma_B^2(N^2-\sum_i N_{i\bullet}^2)(N^2-\sum_j N_{\bullet j}^2).$$

12.6.5 U_e^2 Term 5

$$\begin{aligned}&\frac{1}{4}\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{D}_{A,ii'}\mathbb{D}_{E,rs,r's'}\\&=\frac{1}{4}\Big(\sum_{ii'jj'} Z_{ij}Z_{i'j'}\mathbb{D}_{A,ii'}\Big)\Big(\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{E,rs,r's'}\Big).\end{aligned}$$

The first factor is computed in the previous section. The second factor is

$$\begin{aligned}\sum_{rr'ss'} Z_{rs}Z_{r's'}\mathbb{D}_{E,rs,r's'} &= 2\sigma_E^2\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{rs=r's'})\\&=2\sigma_E^2(N^2-\sum_{rs} Z_{rs})\\&=2\sigma_E^2 N(N-1).\end{aligned}$$

Thus, term 5 is

$$\sigma_A^2\sigma_E^2 N(N-1)(N^2-\sum_i N_{i\bullet}^2).$$

12.6.6 U_e^2 Term 6

By symmetry of indices, this is the same as Term 4:

$$\sigma_A^2\sigma_B^2(N^2-\sum_i N_{i\bullet}^2)(N^2-\sum_j N_{\bullet j}^2).$$

12.6.7 U_e^2 Term 7

This is like term 5 with factors A and B interchanged. Thus, term 7 is equal to

$$\sigma_B^2\sigma_E^2 N(N-1)(N^2 - \sum_j N_{\bullet j}^2).$$

12.6.8 U_e^2 Term 8

By symmetry of indices, this is the same as term 5:

$$\sigma_A^2\sigma_E^2 N(N-1)(N^2 - \sum_i N_{i\bullet}^2).$$

12.6.9 U_e^2 Term 9

By symmetry of indices, this is the same as term 7:

$$\sigma_B^2\sigma_E^2 N(N-1)(N^2 - \sum_j N_{\bullet j}^2).$$

12.6.10 U_e^2 Term 10

$$\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{B}_{A,ii',rr'}\mathbb{B}_{B,jj',ss'} \\
&= \sigma_A^2\sigma_B^2 \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\big(1_{i=r} - 1_{i=r'} - 1_{i'=r} + 1_{i'=r'}\big)\big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big) \\
&= \sigma_A^2\sigma_B^2 \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}(1_{i=r}1_{j=s} - 1_{i=r}1_{j=s'} - 1_{i=r}1_{j'=s} + 1_{i=r}1_{j'=s'} \\
&\qquad - 1_{i=r'}1_{j=s} + 1_{i=r'}1_{j=s'} + 1_{i=r'}1_{j'=s} - 1_{i=r'}1_{j'=s'} \\
&\qquad - 1_{i'=r}1_{j=s} + 1_{i'=r}1_{j=s'} + 1_{i'=r}1_{j'=s} - 1_{i'=r}1_{j'=s'} \\
&\qquad + 1_{i'=r'}1_{j=s} - 1_{i'=r'}1_{j=s'} - 1_{i'=r'}1_{j'=s} + 1_{i'=r'}1_{j'=s'}) \\
&= \sigma_A^2\sigma_B^2\Big(\sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'} - \sum_{ii'jj'}\sum_{r's} Z_{ij}Z_{i'j'}Z_{is}Z_{r'j} - \sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{ij'}Z_{r's'} + \sum_{ii'jj'}\sum_{r's} Z_{ij}Z_{i'j'}Z_{is}Z_{r'j'} \\
&\qquad - \sum_{ii'jj'}\sum_{rs'} Z_{ij}Z_{i'j'}Z_{rj}Z_{is'} + \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs} + \sum_{ii'jj'}\sum_{rs'} Z_{ij}Z_{i'j'}Z_{rj'}Z_{is'} - \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}Z_{ij'} \\
&\qquad - \sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{i'j}Z_{r's'} + \sum_{ii'jj'}\sum_{r's} Z_{ij}Z_{i'j'}Z_{i's}Z_{r'j} + \sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'} - \sum_{ii'jj'}\sum_{r's} Z_{ij}Z_{i'j'}Z_{i's}Z_{r'j'} \\
&\qquad + \sum_{ii'jj'}\sum_{rs'} Z_{ij}Z_{i'j'}Z_{rj}Z_{i's'} - \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}Z_{i'j} - \sum_{ii'jj'}\sum_{rs'} Z_{ij}Z_{i'j'}Z_{rj'}Z_{i's'} + \sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}\Big) \\
&= \sigma_A^2\sigma_B^2\Big(N^3 - N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} - N\sum_{ij'} N_{i\bullet}N_{\bullet j'}Z_{ij'} + \sum_{ij'} N_{i\bullet}^2 N_{\bullet j'}^2 \\
&\qquad - N\sum_{ij} Z_{ij}N_{\bullet j}N_{i\bullet} + N^3 + \sum_{ij'} N_{\bullet j'}^2 N_{i\bullet}^2 - N\sum_{ij'} N_{\bullet j'}N_{i\bullet}Z_{ij'} \\
&\qquad - N\sum_{i'j} N_{\bullet j}N_{i'\bullet}Z_{i'j} + \sum_{i'j} N_{i'\bullet}^2 N_{\bullet j}^2 + N^3 - N\sum_{i'j'} Z_{i'j'}N_{i'\bullet}N_{\bullet j'} \\
&\qquad + \sum_{i'j} N_{\bullet j}^2 N_{i'\bullet}^2 - N\sum_{i'j} N_{\bullet j}N_{i'\bullet}Z_{i'j} - N\sum_{i'j'} Z_{i'j'}N_{\bullet j'}N_{i'\bullet} + N^3\Big) \\
&= 4\sigma_A^2\sigma_B^2\Big(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Big).
\end{aligned}$$

12.6.11 U_e^2 Term 11

$$
\begin{aligned}
&\sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\mathbb{B}_{A,ii',rr'}\mathbb{B}_{E,iji'j',rsr's'}\\
&= \sigma_A^2\sigma_E^2 \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\big(1_{i=r}-1_{i=r'}-1_{i'=r}+1_{i'=r'}\big)\big(1_{ij=rs}-1_{ij=r's'}-1_{i'j'=rs}+1_{i'j'=r's'}\big)\\
&= \sigma_A^2\sigma_E^2 \sum_{ii'jj'}\sum_{rr'ss'} Z_{ij}Z_{i'j'}Z_{rs}Z_{r's'}\big(1_{ij=rs}-1_{i=r}1_{ij=r's'}-1_{i=r}1_{i'j'=rs}+1_{i=r}1_{i'j'=r's'}\\
&\qquad -1_{i=r'}1_{ij=rs}+1_{ij=r's'}+1_{i=r'}1_{i'j'=rs}-1_{i=r'}1_{i'j'=r's'}\\
&\qquad -1_{i'=r}1_{ij=rs}+1_{i'=r}1_{ij=r's'}+1_{i'j'=rs}-1_{i'=r}1_{i'j'=r's'}\\
&\qquad +1_{i'=r'}1_{ij=rs}-1_{i'=r'}1_{ij=r's'}-1_{i'=r'}1_{i'j'=rs}+1_{i'j'=r's'}\big)\\
&= \sigma_A^2\sigma_E^2\big(\sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'}-\sum_{ii'jj'}\sum_{s} Z_{ij}Z_{i'j'}Z_{is}-\sum_{ijj'}\sum_{r's'} Z_{ij}Z_{ij'}Z_{r's'}+\sum_{ii'jj'}\sum_{s} Z_{ij}Z_{i'j'}Z_{is}\\
&\qquad -\sum_{ii'jj'}\sum_{s'} Z_{ij}Z_{i'j'}Z_{is'}+\sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}+\sum_{ii'jj'}\sum_{s'} Z_{ij}Z_{i'j'}Z_{is'}-\sum_{ijj'}\sum_{rs} Z_{ij}Z_{ij'}Z_{rs}\\
&\qquad -\sum_{ijj'}\sum_{r's'} Z_{ij}Z_{ij'}Z_{r's'}+\sum_{ii'jj'}\sum_{s} Z_{ij}Z_{i'j'}Z_{i's}+\sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'}-\sum_{ii'jj'}\sum_{s} Z_{ij}Z_{i'j'}Z_{i's}\\
&\qquad +\sum_{ii'jj'}\sum_{s'} Z_{ij}Z_{i'j'}Z_{i's'}-\sum_{ijj'}\sum_{rs} Z_{ij}Z_{ij'}Z_{rs}-\sum_{ii'jj'}\sum_{s'} Z_{ij}Z_{i'j'}Z_{i's'}+\sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}\big)\\
&= \sigma_A^2\sigma_E^2\big(2\sum_{ii'jj'}\sum_{r's'} Z_{ij}Z_{i'j'}Z_{r's'}-2\sum_{ijj'}\sum_{r's'} Z_{ij}Z_{ij'}Z_{r's'}+2\sum_{ii'jj'}\sum_{rs} Z_{ij}Z_{i'j'}Z_{rs}-2\sum_{ijj'}\sum_{rs} Z_{ij}Z_{ij'}Z_{rs}\big)\\
&= \sigma_A^2\sigma_E^2\big(4N^3-4N\sum_i N_{i\bullet}^2\big).
\end{aligned}
$$

12.6.12 U_e^2 Term 12

We can use the symmetry with term 11, interchanging rows columns. Thus, term 12 is

$$\sigma_B^2\sigma_E^2\big(4N^3-4N\sum_j N_{\bullet j}^2\big).$$

12.7 Combination

Summing up the results of the previous twelve sections, we have

$$
\begin{aligned}
\mathbb{E}(U_e^2) &= \sigma_A^4N^4-2\sigma_A^4N^2\sum_i N_{i\bullet}^2+3\sigma_A^4\Big(\sum_i N_{i\bullet}^2\Big)^2-2\sigma_A^4\sum_i N_{i\bullet}^4+\sigma_A^4(\kappa_A+2)\Big(N^2\sum_i N_{i\bullet}^2-2N\sum_i N_{i\bullet}^3+\sum_i N_{i\bullet}^4\Big)\\
&+\sigma_B^4N^4-2\sigma_B^4N^2\sum_j N_{\bullet j}^2+3\sigma_B^4\Big(\sum_j N_{\bullet j}^2\Big)^2-2\sigma_B^4\sum_j N_{\bullet j}^4+\sigma_B^4(\kappa_B+2)\Big(N^2\sum_j N_{\bullet j}^2-2N\sum_j N_{\bullet j}^3+\sum_j N_{\bullet j}^4\Big)\\
&+\sigma_E^4\Big(N^4-2N^3+3N^2-2N\Big)+\sigma_E^4(\kappa_E+2)N(N-1)^2+\sigma_A^2\sigma_B^2(N^2-\sum_i N_{i\bullet}^2)(N^2-\sum_j N_{\bullet j}^2)\\
&+\sigma_A^2\sigma_E^2N(N-1)(N^2-\sum_i N_{i\bullet}^2)+\sigma_A^2\sigma_B^2(N^2-\sum_i N_{i\bullet}^2)(N^2-\sum_j N_{\bullet j}^2)\\
&+\sigma_B^2\sigma_E^2N(N-1)(N^2-\sum_j N_{\bullet j}^2)+\sigma_A^2\sigma_E^2N(N-1)(N^2-\sum_i N_{i\bullet}^2)+\sigma_B^2\sigma_E^2N(N-1)(N^2-\sum_j N_{\bullet j}^2)\\
&+4\sigma_A^2\sigma_B^2\Big(N^3-2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j}+\sum_{ij} N_{i\bullet}^2N_{\bullet j}^2\Big)+4\sigma_A^2\sigma_E^2\big(N^3-N\sum_i N_{i\bullet}^2\big)+\sigma_B^2\sigma_E^2\big(4N^3-4N\sum_j N_{\bullet j}^2\big).
\end{aligned}
$$

Then, we have

$$\mathrm{Var}(U_e)=\mathbb{E}(U_e^2)-\mathbb{E}(U_e)^2$$

$$
\begin{aligned}
&= \mathbb{E}(U_e^2) - \sigma_A^4(N^2 - \sum_i N_{i\bullet}^2)^2 - \sigma_B^4(N^2 - \sum_j N_{\bullet j}^2)^2 - \sigma_E^4(N^2 - N)^2 \\
&\quad - 2\sigma_A^2\sigma_B^2(N^2 - \sum_i N_{i\bullet}^2)(N^2 - \sum_j N_{\bullet j}^2) - 2\sigma_A^2\sigma_E^2 N(N-1)(N^2 - \sum_i N_{i\bullet}^2) \\
&\quad - 2\sigma_B^2\sigma_E^2 N(N-1)(N^2 - \sum_j N_{\bullet j}^2) \\
&= 2\sigma_A^4\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - 2\sigma_A^4\sum_i N_{i\bullet}^4 + \sigma_A^4(\kappa_A+2)\Bigl(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Bigr) \\
&\quad + 2\sigma_B^4\Bigl(\sum_j N_{\bullet j}^2\Bigr)^2 - 2\sigma_B^4\sum_j N_{\bullet j}^4 + \sigma_B^4(\kappa_B+2)\Bigl(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Bigr) \\
&\quad + 2\sigma_E^4(N^2 - N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_B^2\Bigl(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Bigr) \\
&\quad + 4\sigma_A^2\sigma_E^2\bigl(N^3 - N\sum_i N_{i\bullet}^2\bigr) + \sigma_B^2\sigma_E^2\bigl(4N^3 - 4N\sum_j N_{\bullet j}^2\bigr).
\end{aligned}
$$

Next, we simplify the form of this expression. The coefficient of $(\kappa_A+2)\sigma_A^4$ is

$$
N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4 = \sum_i N_{i\bullet}^2(N - N_{i\bullet})^2
$$

and similarly for that of $(\kappa_B+2)\sigma_B^4$. The coefficient of $(\kappa_E+2)\sigma_E^4$ is $N(N-1)^2$. The remaining multiple of σ_A^4 is

$$
2\Bigl((\sum_i N_{i\bullet}^2)^2 - \sum_i N_{i\bullet}^4\Bigr)
$$

and similarly for σ_B^4. The remaining multiple of σ_E^4 is $2N(N-1)$. The coefficient of $\sigma_A^2\sigma_B^2$ is

$$
4\Bigl(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Bigr) = 4\sum_{ij}(N_{i\bullet}^2 N_{\bullet j}^2 - 2NZ_{ij}N_{i\bullet}N_{\bullet j} + N^2 Z_{ij})
$$

because $N^2\sum_{ij} Z_{ij} = N^3$. Therefore the coefficient of $\sigma_A^2\sigma_B^2$ is

$$
4\sum_{ij}(N_{i\bullet}N_{\bullet j} - NZ_{ij})^2.
$$

Applying these simplifications

$$
\begin{aligned}
\mathrm{Var}(U_e) &= 2\sigma_A^4\Bigl(\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - \sum_i N_{i\bullet}^4\Bigr) + 2\sigma_B^4\Bigl(\Bigl(\sum_j N_{\bullet j}^2\Bigr)^2 - \sum_j N_{\bullet j}^4\Bigr) + 2\sigma_E^4 N(N-1) \\
&\quad + (\kappa_A+2)\sigma_A^4\sum_i N_{i\bullet}^2(N - N_{i\bullet})^2 + (\kappa_B+2)\sigma_B^4\sum_j N_{\bullet j}^2(N - N_{\bullet j})^2 + (\kappa_E+2)\sigma_E^4 N(N-1)^2 \\
&\quad + 4\sigma_A^2\sigma_B^2\sum_{ij}(N_{i\bullet}N_{\bullet j} - NZ_{ij})^2 + 4\sigma_A^2\sigma_E^2 N\bigl(N^2 - \sum_i N_{i\bullet}^2\bigr) + 4\sigma_B^2\sigma_E^2 N\bigl(N^2 - \sum_j N_{\bullet j}^2\bigr).
\end{aligned}
\tag{113}
$$

The coefficient of $\sigma_A^2\sigma_B^2$ is a measure of how close to a regular $R \times C$ grid the data are.

12.8 Check

12.8.1 $\sigma_A^2 = \sigma_B^2 = 0$

If $\sigma_A^2 = \sigma_B^2 = 0$ then Y_{ij} are IID with variance σ_E^2 and kurtosis κ_E. Then

$$
U_e = \frac{1}{2}\sum_{iji'j'}(Y_{ij} - Y_{i'j'})^2 = N(N-1)s_e^2
$$

where s_e is the usual sample standard deviation applied to all N of the Y_{ij}. Thus

$$\mathrm{Var}(U_e) = \sigma_E^4 N^2(N-1)^2\Bigl(\frac{2}{N-1} + \frac{\kappa_E}{N}\Bigr) = \sigma_E^4\bigl(2N^2(N-1) + \kappa_E N(N-1)^2\bigr).$$

Substituting $\sigma_A^2 = \sigma_B^2 = 0$ in $\mathrm{Var}(U_e)$ from Section 12.7 yields

$$2\sigma_E^4(N^2 - N) + \sigma_E^4(\kappa_E + 2)N(N-1)^2 = 2N^2(N-1)\sigma_E^4 + N(N-1)^2\kappa_E\sigma_E^4$$

which matches the formula. Equation (113) becomes

$$\mathrm{Var}(U_e) = 2\sigma_E^4 N(N-1) + (\kappa_E + 2)\sigma_E^4 N(N-1)^2$$

which also matches.

12.8.2 IID sampling

If $\max_i N_{i\bullet} = \max_j N_{\bullet j} = 1$, then the observations are IID with variance $\sigma_Y^2 = \sigma_A^2 + \sigma_B^2 + \sigma_E^2$ and kurtosis

$$\kappa_Y = \frac{\kappa_A\sigma_A^4 + \kappa_B\sigma_B^4 + \kappa_E\sigma_E^4}{\sigma_Y^4}.$$

Now U_e is $N(N-1)$ times the sample standard deviation of all N observations. Thus

$$\begin{aligned}\mathrm{Var}(U_e) &= 2N^2(N-1)\sigma_Y^4 + N(N-1)^2\kappa_Y\sigma_Y^4\\ &= 2N^2(N-1)(\sigma_A^4 + \sigma_B^4 + \sigma_E^4 + 2\sigma_A^2\sigma_B^2 + 2\sigma_A^2\sigma_E^2 + 2\sigma_B^2\sigma_E^2)\\ &\quad + N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_B\sigma_B^4 + \kappa_E\sigma_E^4).\end{aligned} \tag{114}$$

In this case, the formula gives

$$\begin{aligned}&2\sigma_A^4\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - 2\sigma_A^4\sum_i N_{i\bullet}^4 + \sigma_A^4(\kappa_A+2)\Bigl(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Bigr)\\ &\quad + 2\sigma_B^4\Bigl(\sum_j N_{\bullet j}^2\Bigr)^2 - 2\sigma_B^4\sum_j N_{\bullet j}^4 + \sigma_B^4(\kappa_B+2)\Bigl(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Bigr)\\ &\quad + 2\sigma_E^4(N^2-N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_B^2\Bigl(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Bigr)\\ &\quad + 4\sigma_A^2\sigma_E^2\bigl(N^3 - N\sum_i N_{i\bullet}^2\bigr) + \sigma_B^2\sigma_E^2\bigl(4N^3 - 4N\sum_j N_{\bullet j}^2\bigr).\end{aligned}$$

If we set all positive $N_{i\bullet} = 1$ and all positive $N_{\bullet j} = 1$ then $\sum_i N_{i\bullet}^2 = N$ because there are now $R = N$ rows in the data. Similarly $N_{i\bullet}^3$ and $N_{i\bullet}^4$ sum to N and these powers of $N_{\bullet j}$ also sum to N. Next $\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} = \sum_{ij} Z_{ij} = N$. The most subtle of these sums is $\sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2 = \sum_{ij} 1 = N^2$ because the indices run over all i with $N_{i\bullet} > 0$ and all j with $N_{\bullet j} > 0$.

Making these substitutions we get

$$\begin{aligned}\mathrm{Var}(U_e) &= 2\sigma_A^4 N^2 - 2\sigma_A^4 N + \sigma_A^4(\kappa_A+2)(N^3 - 2N^2 + N)\\ &\quad + 2\sigma_B^4 N^2 - 2\sigma_B^4 N + \sigma_B^4(\kappa_B+2)(N^3 - 2N^2 + N)\\ &\quad + 2\sigma_E^4(N^2 - N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_B^2(N^3 - 2N^2 + N^2)\\ &\quad + 4\sigma_A^2\sigma_E^2(N^3 - N^2) + \sigma_B^2\sigma_E^2(4N^3 - 4N^2)\\ &= 2\sigma_A^4\bigl(N^2 - N + N(N-1)^2\bigr)\\ &\quad + 2\sigma_B^4\bigl(N^2 - N + N(N-1)^2\bigr)\\ &\quad + 2\sigma_E^4\bigl(N(N-1) + N(N-1)^2\bigr) + 4\sigma_A^2\sigma_B^2 N^2(N-1)\end{aligned}$$

$$
\begin{aligned}
&+ 4\sigma_A^2\sigma_E^2 N^2(N-1) + 4\sigma_B^2\sigma_E^2 N^2(N-1)\\
&+ N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_B\sigma_B^4 + \kappa_E\sigma_E^4)\\
&= 2N^2(N-1)(\sigma_A^4 + \sigma_B^4 + \sigma_E^4)\\
&+ 4N^2(N-1)\bigl(\sigma_A^2\sigma_B^2 + \sigma_A^2\sigma_E^2 + \sigma_A^2\sigma_E^2\bigr)\\
&+ N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_B\sigma_B^4 + \kappa_E\sigma_E^4)
\end{aligned}
$$

which matches equation (114).

Equation (113) gives

$$
\begin{aligned}
\mathrm{Var}(U_e) &= 2\sigma_A^4\Bigl(\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - \sum_i N_{i\bullet}^4\Bigr) + 2\sigma_B^4\Bigl(\Bigl(\sum_j N_{\bullet j}^2\Bigr)^2 - \sum_j N_{\bullet j}^4\Bigr) + 2\sigma_E^4 N(N-1)\\
&+ (\kappa_A+2)\sigma_A^4\sum_i N_{i\bullet}^2(N-N_{i\bullet})^2 + (\kappa_B+2)\sigma_B^4\sum_j N_{\bullet j}^2(N-N_{\bullet j})^2 + (\kappa_E+2)\sigma_E^4 N(N-1)^2\\
&+ 4\sigma_A^2\sigma_B^2\sum_{ij}(N_{i\bullet}N_{\bullet j} - NZ_{ij})^2 + 4\sigma_A^2\sigma_E^2 N\bigl(N^2 - \sum_i N_{i\bullet}^2\bigr) + 4\sigma_B^2\sigma_E^2 N\bigl(N^2 - \sum_j N_{\bullet j}^2\bigr)\\
&= 2\sigma_A^4(N^2-N) + 2\sigma_B^4(N^2-N) + 2\sigma_E^4 N(N-1)\\
&+ (\kappa_A+2)\sigma_A^4 N(N-1)^2 + (\kappa_B+2)\sigma_B^4 N(N-1)^2 + (\kappa_E+2)\sigma_E^4 N(N-1)^2\\
&+ 4\sigma_A^2\sigma_B^2 N^2(N-1)^2 + 4\sigma_A^2\sigma_E^2 N(N^2-N) + 4\sigma_B^2\sigma_E^2 N(N^2-N).
\end{aligned}
$$

12.8.3 IID sampling again

If $\max_i N_{i\bullet} = 1$ and $\sigma_B^2 = 0$, then once again the observations are IID and

$$
\begin{aligned}
\mathrm{Var}(U_e) &= 2N^2(N-1)\sigma_Y^4 + N(N-1)^2\kappa_Y\sigma_Y^4\\
&= 2N^2(N-1)(\sigma_A^4 + \sigma_B^4 + \sigma_E^4 + 2\sigma_A^2\sigma_B^2 + 2\sigma_A^2\sigma_E^2 + 2\sigma_B^2\sigma_E^2)\\
&\quad + N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_B\sigma_B^4 + \kappa_E\sigma_E^4)\\
&= 2N^2(N-1)(\sigma_A^4 + \sigma_E^4 + 2\sigma_A^2\sigma_E^2) + N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_E\sigma_E^4).
\end{aligned} \tag{115}
$$

The formula gives

$$
\begin{aligned}
\mathrm{Var}(U_e) &= 2\sigma_A^4\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - 2\sigma_A^4\sum_i N_{i\bullet}^4 + \sigma_A^4(\kappa_A+2)\Bigl(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Bigr)\\
&+ 2\sigma_B^4\Bigl(\sum_j N_{\bullet j}^2\Bigr)^2 - 2\sigma_B^4\sum_j N_{\bullet j}^4 + \sigma_B^4(\kappa_B+2)\Bigl(N^2\sum_j N_{\bullet j}^2 - 2N\sum_j N_{\bullet j}^3 + \sum_j N_{\bullet j}^4\Bigr)\\
&+ 2\sigma_E^4(N^2-N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_B^2\Bigl(N^3 - 2N\sum_{ij} Z_{ij}N_{i\bullet}N_{\bullet j} + \sum_{ij} N_{i\bullet}^2 N_{\bullet j}^2\Bigr)\\
&+ 4\sigma_A^2\sigma_E^2\bigl(N^3 - N\sum_i N_{i\bullet}^2\bigr) + \sigma_B^2\sigma_E^2\bigl(4N^3 - 4N\sum_j N_{\bullet j}^2\bigr)\\
&= 2\sigma_A^4\Bigl(\sum_i N_{i\bullet}^2\Bigr)^2 - 2\sigma_A^4\sum_i N_{i\bullet}^4 + \sigma_A^4(\kappa_A+2)\Bigl(N^2\sum_i N_{i\bullet}^2 - 2N\sum_i N_{i\bullet}^3 + \sum_i N_{i\bullet}^4\Bigr)\\
&+ 2\sigma_E^4(N^2-N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_E^2(N^3 - N\sum_i N_{i\bullet}^2)\\
&= 2\sigma_A^4 N^2 - 2\sigma_A^4 N + \sigma_A^4(\kappa_A+2)(N^3 - 2N^2 + N)\\
&+ 2\sigma_E^4(N^2-N) + \sigma_E^4(\kappa_E+2)N(N-1)^2 + 4\sigma_A^2\sigma_E^2(N^3 - N^2)\\
&= 2N^2(N-1)(\sigma_A^4 + \sigma_E^4 + 2\sigma_A^2\sigma_E^2) + N(N-1)^2(\kappa_A\sigma_A^4 + \kappa_E\sigma_E^4),
\end{aligned}
$$

matching (115).

13 Covariance of U_a and U_b

We use the formula $\mathrm{Cov}(U_a, U_b) = \mathbb{E}(U_a U_b) - \mathbb{E}(U_a)\mathbb{E}(U_b)$, so we just need to compute $\mathbb{E}(U_a U_b)$. Using our preferred normalization,

$$\begin{aligned}
U_a U_b &= \frac{1}{4}\big(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (Y_{ij} - Y_{ij'})^2\big)\big(\sum_{rr's} N_{\bullet s}^{-1} Z_{rs} Z_{r's} (Y_{rs} - Y_{r's})^2\big) \\
&= \frac{1}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} (Y_{ij} - Y_{ij'})^2 (Y_{rs} - Y_{r's})^2
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}(U_a U_b) = \frac{1}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} \big(&\underbrace{\mathbb{Q}_{E,ijij',rsr's}}_{\text{Term 1}} \\
&+ \underbrace{\mathbb{D}_{B,jj'} \mathbb{D}_{A,rr'}}_{\text{Term 2}} + \underbrace{\mathbb{D}_{B,jj'} \mathbb{D}_{E,rs,r's}}_{\text{Term 3}} + \underbrace{\mathbb{D}_{E,ij,ij'} \mathbb{D}_{A,rr'}}_{\text{Term 4}} \big).
\end{aligned}$$

We consider each term separately.

13.1 $U_a U_b$ Term 1

$$\begin{aligned}
&\frac{1}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} \mathbb{Q}_{E,ijij',rsr's} \\
&= \frac{\sigma_E^4}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 1_{ij \neq ij'} 1_{rs \neq r's} \\
&\quad \Big(4 + (\kappa_E + 2)(1_{ij \in \{rs, r's\}} + 1_{ij' \in \{rs, r's\}}) + 4 \times 1_{\{ij, ij'\} = \{rs, r's\}}\Big) \\
&= \frac{\sigma_E^4}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 1_{j \neq j'} 1_{r \neq r'} \\
&\quad \Big(\underbrace{4}_{1.1} + \underbrace{(\kappa_E + 2)(1_{ij \in \{rs, r's\}} + 1_{ij' \in \{rs, r's\}})}_{1.2 \text{ and } 1.3} + \underbrace{4 \times 1_{\{ij, ij'\} = \{rs, r's\}}}_{1.4}\Big).
\end{aligned}$$

For 1.1, we have

$$\begin{aligned}
&\sigma_E^4 \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 1_{j \neq j'} 1_{r \neq r'} \\
&= \sigma_E^4 \big(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (1 - 1_{j = j'})\big)\big(\sum_{rr's} N_{\bullet s}^{-1} Z_{rs} Z_{r's} (1 - 1_{r = r'})\big) \\
&= \sigma_E^4 \big(\sum_{i} N_{i\bullet} - \sum_{ij} N_{i\bullet}^{-1} Z_{ij}\big)\big(\sum_{s} N_{\bullet s} - \sum_{rs} N_{\bullet s}^{-1} Z_{rs}\big) \\
&= \sigma_E^4 \big(N - R\big)\big(N - C\big).
\end{aligned}$$

For 1.2, we have $\sigma_E^4(\kappa_E + 2)/4$ times

$$\begin{aligned}
&\sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 1_{j \neq j'} 1_{r \neq r'} 1_{ij \in \{rs, r's\}} \\
&= 2 \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 1_{j \neq j'} 1_{r \neq r'} 1_{ij = rs} \\
&= 2 \sum_{ijj'} \sum_{r'} N_{i\bullet}^{-1} N_{\bullet j}^{-1} Z_{ij} Z_{ij'} Z_{r'j} 1_{j \neq j'} 1_{i \neq r'}
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{ijj'}\sum_{r'} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}Z_{ij'}Z_{r'j}(1-1_{j=j'}-1_{i=r'}+1_{j=j'}1_{i=r'})\\
&= 2\sum_{ijj'} N_{i\bullet}^{-1}Z_{ij}Z_{ij'} - 2\sum_{ij}\sum_{r'} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}Z_{r'j} - 2\sum_{ijj'} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}Z_{ij'} + 2\sum_{ij} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}\\
&= 2\sum_{ij} Z_{ij} - 2\sum_{ij} N_{i\bullet}^{-1}Z_{ij} - 2\sum_{ij} N_{\bullet j}^{-1}Z_{ij} + 2\sum_{ij} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}\\
&= 2\sum_{ij} Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1}).
\end{aligned}
$$

Term 1.3 is the same as 1.2 by symmetry of indices.

For 1.4, we have

$$
\sigma_E^4\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}1_{j\neq j'}1_{r\neq r'}1_{\{ij,ij'\}=\{rs,r's\}} = 0,
$$

since the last indicator implies $r=i$ and $r'=i$ but the second one is $1_{r\neq r'}$.

Summing up, term 1 is equal to

$$
\begin{aligned}
&\sigma_E^4\big(N-R\big)\big(N-C\big) + \sigma_E^4(\kappa_E+2)\sum_{ij} Z_{ij}(1-N_{i\bullet}^{-1})(1-N_{\bullet j}^{-1})\\
&= \sigma_E^4\big(N-R\big)\big(N-C\big) + \sigma_E^4(\kappa_E+2)(N-R-C+\sum_{ij} N_{i\bullet}^{-1}N_{\bullet j}^{-1}Z_{ij}).
\end{aligned}
$$

13.2 U_aU_b Term 2

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}2\sigma_B^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})\\
&= \sigma_A^2\sigma_B^2\big(\sum_{ijj'} N_{i\bullet}^{-1}Z_{ij}Z_{ij'}(1-1_{j=j'})\big)\big(\sum_{rr's} N_{\bullet s}^{-1}Z_{rs}Z_{r's}(1-1_{r=r'})\big)\\
&= \sigma_A^2\sigma_B^2\big(\sum_{i} N_{i\bullet} - \sum_{ij} N_{i\bullet}^{-1}Z_{ij}\big)\big(\sum_{s} N_{\bullet s} - \sum_{rs} N_{\bullet s}^{-1}Z_{rs}\big)\\
&= \sigma_A^2\sigma_B^2\big(N-R\big)\big(N-C\big).
\end{aligned}
$$

13.3 U_aU_b Term 3

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}2\sigma_B^2(1-1_{j=j'})2\sigma_E^2(1-1_{r=r'})\\
&= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N-C\big)
\end{aligned}
$$

using the previous section.

13.4 U_aU_b Term 4

$$
\frac{1}{4}\sum_{ijj'}\sum_{rr's} N_{i\bullet}^{-1}N_{\bullet s}^{-1}Z_{ij}Z_{ij'}Z_{rs}Z_{r's}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}
$$

$$= \frac{1}{4} \sum_{ijj'} \sum_{rr's} N_{i\bullet}^{-1} N_{\bullet s}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's} 2\sigma_E^2 (1 - 1_{j=j'}) 2\sigma_A^2 (1 - 1_{r=r'})$$
$$= \sigma_A^2 \sigma_E^2 \big(N - R\big)\big(N - C\big)$$

using the previous section.

13.5 Combination

Adding up the four terms, we have

$$\mathbb{E}(U_a U_b) = \sigma_E^4 \big(N - R\big)\big(N - C\big) + \sigma_E^4 (\kappa_E + 2) \sum_{ij} Z_{ij} (1 - N_{i\bullet}^{-1})(1 - N_{\bullet j}^{-1})$$
$$+ \sigma_A^2 \sigma_B^2 \big(N - R\big)\big(N - C\big) + \sigma_B^2 \sigma_E^2 \big(N - R\big)\big(N - C\big) + \sigma_A^2 \sigma_E^2 \big(N - R\big)\big(N - C\big),$$

and so

$$\begin{aligned} \mathrm{Cov}(U_a, U_b) &= \mathbb{E}(U_a U_b) - \mathbb{E}(U_a)\mathbb{E}(U_b) \\ &= \mathbb{E}(U_a U_b) - (\sigma_B^2 + \sigma_E^2)(\sigma_A^2 + \sigma_E^2)(N - R)(N - C) \\ &= \sigma_E^4 (\kappa_E + 2) \sum_{ij} Z_{ij} (1 - N_{i\bullet}^{-1})(1 - N_{\bullet j}^{-1}). \end{aligned}$$

Notice that $\mathrm{Cov}(U_a, U_b) = 0$ when $\sigma_E^2 = 0$. This can be verified by noting that when $\sigma_E^2 = 0$ then U_a is a function only of a_i while U_b is a function only of b_j. Therefore U_a and U_b are independent when $\sigma_E^2 = 0$.

14 Covariance of U_a and U_e

We use the formula $\mathrm{Cov}(U_a, U_e) = \mathbb{E}(U_a U_e) - \mathbb{E}(U_a)\mathbb{E}(U_e)$, so we just need to compute $\mathbb{E}(U_a U_e)$. First,

$$\begin{aligned} U_a U_e &= \frac{1}{4} \Big(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (Y_{ij} - Y_{ij'})^2 \Big) \Big(\sum_{rr'ss'} Z_{rs} Z_{r's'} (Y_{rs} - Y_{r's'})^2 \Big) \\ &= \frac{1}{4} \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} (Y_{ij} - Y_{ij'})^2 (Y_{rs} - Y_{r's'})^2. \end{aligned}$$

Then,

$$\mathbb{E}(U_a U_e) = \frac{1}{4} \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} \Big(\underbrace{\mathbb{Q}_{B,jj',ss'}}_{\text{Term 1}} + \underbrace{\mathbb{Q}_{E,ijij',rsr's'}}_{\text{Term 2}} + \underbrace{\mathbb{D}_{B,jj'} \mathbb{D}_{A,rr'}}_{\text{Term 3}} + \underbrace{\mathbb{D}_{B,jj'} \mathbb{D}_{E,rs,r's'}}_{\text{Term 4}}$$
$$+ \underbrace{\mathbb{D}_{E,ij,ij'} \mathbb{D}_{A,rr'}}_{\text{Term 5}} + \underbrace{\mathbb{D}_{E,ij,ij'} \mathbb{D}_{B,ss'}}_{\text{Term 6}} + \underbrace{4\mathbb{B}_{B,jj',ss'} \mathbb{B}_{E,ijij',rsr's'}}_{\text{Term 7}} \Big).$$

We consider each term separately.

14.1 $U_a U_e$ Term 1

$$\frac{1}{4} \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} \mathbb{Q}_{B,jj',ss'}$$
$$= \frac{1}{4} \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j \neq j'} 1_{s \neq s'} \sigma_B^4 \Big(\underbrace{4}_{1.1} + \underbrace{(\kappa_B + 2)(1_{j \in \{s,s'\}} + 1_{j' \in \{s,s'\}})}_{\text{1.2 and 1.3}} + \underbrace{4 \times 1_{\{j,j'\} = \{s,s'\}}}_{1.4} \Big).$$

Term 1.1 is equal to σ_B^4 times

$$\sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j \neq j'} 1_{s \neq s'}$$

$$
\begin{aligned}
&= \Big(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (1 - 1_{j=j'})\Big)\Big(\sum_{rr'ss'} Z_{rs} Z_{r's'} (1 - 1_{s=s'})\Big) \\
&= \Big(\sum_{i} N_{i\bullet} - \sum_{ij} N_{i\bullet}^{-1} Z_{ij}\Big)\Big(N^2 - \sum_{rr's} Z_{rs} Z_{r's}\Big) \\
&= \big(N - R\big)\big(N^2 - \sum_{s} N_{\bullet s}^2\big).
\end{aligned}
$$

Term 1.2 is equal to $\sigma_B^4(\kappa_B + 2)/4$ times

$$
\begin{aligned}
&\sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{s\neq s'} 1_{j\in\{s,s'\}} \\
&= 2 \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{s\neq s'} 1_{j=s} \\
&= 2 \sum_{ijj'} \sum_{rr's'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{r's'} (1 - 1_{j=j'})(1 - 1_{j=s'}) \\
&= 2 \sum_{ijj'} \sum_{rr's'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{r's'} - 2 \sum_{ij} \sum_{rr's'} N_{i\bullet}^{-1} Z_{ij} Z_{rj} Z_{r's'} \\
&\quad - 2 \sum_{ijj'} \sum_{rr'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{r'j} + 2 \sum_{ij} \sum_{rr'} N_{i\bullet}^{-1} Z_{ij} Z_{rj} Z_{r'j} \\
&= 2N \sum_{ijj'} \sum_{r} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} - 2N \sum_{ij} \sum_{r} N_{i\bullet}^{-1} Z_{ij} Z_{rj} \\
&\quad - 2 \sum_{ijj'} \sum_{r} N_{i\bullet}^{-1} N_{\bullet j} Z_{ij} Z_{ij'} Z_{rj} + 2 \sum_{ij} \sum_{r} N_{i\bullet}^{-1} N_{\bullet j} Z_{ij} Z_{rj} \\
&= 2N \sum_{ij} Z_{ij} N_{\bullet j} - 2N \sum_{ij} N_{i\bullet}^{-1} Z_{ij} N_{\bullet j} - 2 \sum_{ij} Z_{ij} N_{\bullet j}^2 + 2 \sum_{ij} N_{i\bullet}^{-1} Z_{ij} N_{\bullet j}^2 \\
&= 2 \sum_{ij} Z_{ij} (N N_{\bullet j} - N N_{i\bullet}^{-1} N_{\bullet j} - N_{\bullet j}^2 + N_{i\bullet}^{-1} N_{\bullet j}^2) \\
&= 2 \sum_{ij} Z_{ij} (N - N_{\bullet j}) N_{\bullet j} (1 - N_{i\bullet}^{-1}).
\end{aligned}
$$

Term 1.3 is equal to term 1.2 by symmetry of indices.

Term 1.4 is equal to σ_B^4 times

$$
\begin{aligned}
&\sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{s\neq s'} 1_{\{j,j'\}=\{s,s'\}} \\
&= 2 \sum_{ijj'} \sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{s\neq s'} 1_{j=s} 1_{j'=s'} \\
&= 2 \sum_{ijj'} \sum_{rr'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rj} Z_{r'j'} (1 - 1_{j=j'}) \\
&= 2 \sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} N_{\bullet j} N_{\bullet j'} - 2 \sum_{ij} \sum_{rr'} N_{i\bullet}^{-1} Z_{ij} Z_{rj} Z_{r'j} \\
&= 2 \sum_{i} N_{i\bullet}^{-1} \Big(\sum_{j} Z_{ij} N_{\bullet j}\Big)^2 - 2 \sum_{ij} N_{i\bullet}^{-1} Z_{ij} N_{\bullet j}^2.
\end{aligned}
$$

Summing the four terms, we find that term 1 is equal to

$$
\begin{aligned}
&\sigma_B^4 \big(N - R\big)\big(N^2 - \sum_{j} N_{\bullet j}^2\big) + 2\sigma_B^4 \Big(\sum_{i} N_{i\bullet}^{-1} \Big(\sum_{j} Z_{ij} N_{\bullet j}\Big)^2 - \sum_{ij} N_{i\bullet}^{-1} Z_{ij} N_{\bullet j}^2\Big) \\
&+ \sigma_B^4 (\kappa_B + 2) \sum_{ij} Z_{ij} (N - N_{\bullet j}) N_{\bullet j} (1 - N_{i\bullet}^{-1}).
\end{aligned}
$$

14.2 $U_a U_e$ Term 2

$$\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} \mathbb{Q}_{E,ijij',rsr's'}$$
$$= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'} \sigma_E^4$$
$$\Big(\underbrace{4}_{2.1} + \underbrace{(\kappa_E+2)(1_{ij\in\{rs,r's'\}} + 1_{ij'\in\{rs,r's'\}})}_{2.2 \text{ and } 2.3} + \underbrace{4\times 1_{\{ij,ij'\}=\{rs,r's'\}}}_{2.4}\Big).$$

For 2.1, we get σ_E^4 times

$$\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'}$$
$$= N(N-1)\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} (1 - 1_{j=j'})$$
$$= N(N-1)\big(\sum_i N_{i\bullet} - \sum_{ij} N_{i\bullet}^{-1} Z_{ij}\big)$$
$$= N(N-1)(N-R).$$

For 2.2, we get $\sigma_E^4(\kappa_E+2)/4$ times

$$\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'} 1_{ij\in\{rs,r's'\}}$$
$$= 2\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'} 1_{ij=rs}$$
$$= 2\sum_{ijj'}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{r's'} (1 - 1_{j=j'})(1 - 1_{ij=r's'})$$
$$= 2\sum_{ijj'}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{r's'} (1 - 1_{j=j'} - 1_{ij=r's'} + 1_{j=j'} 1_{ij=r's'})$$
$$= 2N\sum_i N_{i\bullet} - 2N\sum_{ij} N_{i\bullet}^{-1} Z_{ij} - 2\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} + 2\sum_{ij} N_{i\bullet}^{-1} Z_{ij}$$
$$= 2N^2 - 2NR - 2N + 2R$$
$$= 2(N-R)(N-1).$$

Term 2.3 is the same as 2.2 by symmetry of indices.

For term 2.4, we get σ_E^4 times

$$\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'} 1_{\{ij,ij'\}=\{rs,r's'\}}$$
$$= 2\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} Z_{rs} Z_{r's'} 1_{j\neq j'} 1_{rs\neq r's'} 1_{ij=rs} 1_{ij'=r's'}$$
$$= 2\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij} Z_{ij'} 1_{j\neq j'}$$
$$= 2\sum_i N_{i\bullet} - 2\sum_{ij} N_{i\bullet}^{-1} Z_{ij}$$
$$= 2(N-R).$$

Adding up the four terms, we find that term 2 equals

$$\sigma_E^4 N(N-1)(N-R) + 2\sigma_E^4(N-R) + \sigma_E^4(\kappa_E+2)(N-R)(N-1).$$

14.3 U_aU_e Term 3

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{B,jj'}\mathbb{D}_{A,rr'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_B^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})\\
&= \sigma_A^2\sigma_B^2\bigl(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(1-1_{j=j'})\bigr)\bigl(\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{r=r'})\bigr)\\
&= \sigma_A^2\sigma_B^2\bigl(\sum_{i} N_{i\bullet} - \sum_{ij} N_{i\bullet}^{-1} Z_{ij}\bigr)\bigl(N^2 - \sum_{rss'} Z_{rs}Z_{rs'}\bigr)\\
&= \sigma_A^2\sigma_B^2\bigl(N-R\bigr)\bigl(N^2 - \sum_{r} N_{r\bullet}^2\bigr).
\end{aligned}
$$

14.4 U_aU_e Term 4

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{B,jj'}\mathbb{D}_{E,rs,r's'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_B^2(1-1_{j=j'})2\sigma_E^2(1-1_{r=r'}1_{s=s'})\\
&= \sigma_B^2\sigma_E^2\bigl(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(1-1_{j=j'})\bigr)\bigl(\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{r=r'}1_{s=s'})\bigr)\\
&= \sigma_B^2\sigma_E^2\bigl(N-R\bigr)\bigl(N^2 - \sum_{rs} Z_{rs}\bigr)\\
&= \sigma_B^2\sigma_E^2\bigl(N-R\bigr)\bigl(N^2 - N\bigr).
\end{aligned}
$$

14.5 U_aU_e Term 5

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{A,rr'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_E^2(1-1_{j=j'})2\sigma_A^2(1-1_{r=r'})\\
&= \sigma_A^2\sigma_E^2\bigl(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(1-1_{j=j'})\bigr)\bigl(\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{r=r'})\bigr)\\
&= \sigma_A^2\sigma_E^2\bigl(N-R\bigr)\bigl(N^2 - \sum_{r} N_{r\bullet}^2\bigr)
\end{aligned}
$$

using the result for term 3.

14.6 U_aU_e Term 6

$$
\begin{aligned}
&\frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{D}_{E,ij,ij'}\mathbb{D}_{B,ss'}\\
&= \frac{1}{4}\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}2\sigma_E^2(1-1_{j=j'})2\sigma_B^2(1-1_{s=s'})\\
&= \sigma_B^2\sigma_E^2\bigl(\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}(1-1_{j=j'})\bigr)\bigl(\sum_{rr'ss'} Z_{rs}Z_{r's'}(1-1_{s=s'})\bigr)
\end{aligned}
$$

$$= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2 - \sum_{rr's} Z_{rs}Z_{r's}\big)$$
$$= \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2 - \sum_{s} N_{\bullet s}^2\big).$$

14.7 U_aU_e Term 7

$$\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\mathbb{B}_{B,jj',ss'}\mathbb{B}_{E,ijij',rsr's'}$$
$$= \sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\sigma_B^2\big(1_{j=s} - 1_{j=s'} - 1_{j'=s} + 1_{j'=s'}\big)$$
$$\sigma_E^2\big(1_{ij=rs} - 1_{ij=r's'} - 1_{ij'=rs} + 1_{ij'=r's'}\big)$$
$$\begin{aligned}= \sigma_B^2\sigma_E^2\sum_{ijj'}\sum_{rr'ss'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}Z_{r's'}\big(\quad & 1_{ij=rs} - 1_{j=s}1_{ij=r's'} - 1_{j=s}1_{ij'=rs} + 1_{j=s}1_{ij'=r's'}\\ & - 1_{j=s'}1_{ij=rs} + 1_{ij=r's'} + 1_{j=s'}1_{ij'=rs} - 1_{j=s'}1_{ij'=r's'}\\ & - 1_{j'=s}1_{ij=rs} + 1_{j'=s}1_{ij=r's'} + 1_{ij'=rs} - 1_{j'=s}1_{ij'=r's'}\\ & + 1_{j'=s'}1_{ij=rs} - 1_{j'=s'}1_{ij=r's'} - 1_{j'=s'}1_{ij'=rs} + 1_{ij'=r's'}\big)\end{aligned}$$
$$\begin{aligned}= \sigma_B^2\sigma_E^2\big(&\sum_{ijj'}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r's'} - \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rj} - \sum_{ij}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij}Z_{r's'} + \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rj}\\ & - \sum_{ijj'}\sum_{r'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r'j} + \sum_{ijj'}\sum_{rs} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs} + \sum_{ijj'}\sum_{r'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r'j} - \sum_{ij}\sum_{rs} N_{i\bullet}^{-1} Z_{ij}Z_{rs}\\ & - \sum_{ij}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij}Z_{r's'} + \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rj'} + \sum_{ijj'}\sum_{r's'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r's'} - \sum_{ijj'}\sum_{r} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rj'}\\ & + \sum_{ijj'}\sum_{r'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r'j'} - \sum_{ij}\sum_{rs} N_{i\bullet}^{-1} Z_{ij}Z_{rs} - \sum_{ijj'}\sum_{r'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{r'j'} + \sum_{ijj'}\sum_{rs} N_{i\bullet}^{-1} Z_{ij}Z_{ij'}Z_{rs}\big)\end{aligned}$$
$$= 4\sigma_B^2\sigma_E^2\big(N\sum_{ijj'} N_{i\bullet}^{-1} Z_{ij}Z_{ij'} - N\sum_{ij} N_{i\bullet}^{-1} Z_{ij}\big)$$
$$= 4\sigma_B^2\sigma_E^2 N\big(\sum_i N_{i\bullet} - \sum_i 1\big)$$
$$= 4\sigma_B^2\sigma_E^2 N(N-R).$$

14.8 Combination

We add up the seven terms, replacing some $N_{r\bullet}$ and $N_{\bullet s}$ expressions by equivalents using $N_{i\bullet}$ and $N_{\bullet j}$, getting

$$\begin{aligned}\mathbb{E}(U_aU_e) = {} & \sigma_B^4\big(N-R\big)\big(N^2 - \sum_j N_{\bullet j}^2\big) + 2\sigma_B^4\Big(\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij}N_{\bullet j}\Big)^2 - \sum_{ij} N_{i\bullet}^{-1} Z_{ij}N_{\bullet j}^2\Big)\\ & + \sigma_B^4(\kappa_B + 2)\sum_{ij} Z_{ij}(N - N_{\bullet j})N_{\bullet j}(1 - N_{i\bullet}^{-1})\\ & + \sigma_E^4 N(N-1)(N-R) + 2\sigma_E^4(N-R) + \sigma_E^4(\kappa_E+2)(N-R)(N-1)\\ & + \sigma_A^2\sigma_B^2\big(N-R\big)\big(N^2 - \sum_i N_{i\bullet}^2\big) + \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2 - N\big)\\ & + \sigma_A^2\sigma_E^2\big(N-R\big)\big(N^2 - \sum_i N_{i\bullet}^2\big) + \sigma_B^2\sigma_E^2\big(N-R\big)\big(N^2 - \sum_j N_{\bullet j}^2\big)\\ & + 4\sigma_B^2\sigma_E^2 N(N-R).\end{aligned}$$

Now

$$\mathbb{E}(U_a)\mathbb{E}(U_e) = (N-R)(\sigma_B^2+\sigma_E^2)\Big(\sigma_A^2(N^2-\sum_i N_{i\bullet}^2)+\sigma_B^2(N^2-\sum_j N_{\bullet j}^2)+\sigma_E^2(N^2-N)\Big)$$

which contains terms equalling several of those in $\mathbb{E}(U_aU_e)$ above. Subtracting those term from $\mathbb{E}(U_aU_e)$ yields

$$\begin{aligned}\mathrm{Cov}(U_a,U_e) = 2\sigma_B^4&\Big(\sum_i N_{i\bullet}^{-1}\Big(\sum_j Z_{ij}N_{\bullet j}\Big)^2-\sum_{ij}N_{i\bullet}^{-1}Z_{ij}N_{\bullet j}^2\Big)\\&+\sigma_B^4(\kappa_B+2)\sum_{ij}Z_{ij}(N-N_{\bullet j})N_{\bullet j}(1-N_{i\bullet}^{-1})\\&+2\sigma_E^4(N-R)+\sigma_E^4(\kappa_E+2)(N-R)(N-1)\\&+4\sigma_B^2\sigma_E^2N(N-R).\end{aligned}$$

15 Covariance of U_b and U_e

By interchanging the roles of the rows and columns in $\mathrm{Cov}(U_a,U_e)$, we find that

$$\begin{aligned}\mathrm{Cov}(U_b,U_e) = 2\sigma_A^4&\Big(\sum_j N_{\bullet j}^{-1}\Big(\sum_i Z_{ij}N_{i\bullet}\Big)^2-\sum_{ij}N_{\bullet j}^{-1}Z_{ij}N_{i\bullet}^2\Big)\\&+\sigma_A^4(\kappa_A+2)\sum_{ij}Z_{ij}(N-N_{i\bullet})N_{i\bullet}(1-N_{\bullet j}^{-1})\\&+2\sigma_E^4(N-C)+\sigma_E^4(\kappa_E+2)(N-C)(N-1)\\&+4\sigma_A^2\sigma_E^2N(N-C).\end{aligned}$$

16 Asymptotic approximation: proof of Theorem 4.2

We suppose that the following inequalities all hold

$$\begin{array}{llll} N_{i\bullet}\leqslant \epsilon N, & N_{\bullet j}\leqslant\epsilon N, & R\leqslant\epsilon N, & C\leqslant\epsilon N,\\ N\leqslant\epsilon\sum_i N_{i\bullet}^2, & N\leqslant\epsilon\sum_j N_{\bullet j}^2, & \sum_i N_{i\bullet}^2\leqslant\epsilon N^2,\quad\text{and} & \sum_j N_{\bullet j}^2\leqslant\epsilon N^2\end{array}$$

for the same small $\epsilon>0$. The first six inequalities are assumed in the theorem statement. The last two follow from the first two. We also assume that

$$0<\underline{m}\leqslant\kappa_A+2,\kappa_B+2,\kappa_E+2,\sigma_A^4,\sigma_B^4,\sigma_E^4\leqslant\overline{m}<\infty.$$

Note that we can bound $\sigma_A^2\sigma_B^2$, $\sigma_A^2\sigma_E^2$, and $\sigma_A^2\sigma_B^2$ away from 0 and ∞ uniformly with those other quantities after replacing $\underline{m}$ by $\min(\underline{m},\underline{m}^2)$ and $\overline{m}$ by $\max(\overline{m},\overline{m}^2)$.

We also suppose that

$$\sum_{ij}Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}\leqslant\epsilon\sum_i N_{i\bullet}^2,\quad\text{and}\qquad \sum_{ij}Z_{ij}N_{i\bullet}N_{\bullet j}^{-1}\leqslant\epsilon\sum_j N_{\bullet j}^2. \tag{116}$$

The bounds in (116) seem reasonable but it appears that they cannot be derived from the first eight bounds above.

We begin with the coefficient of $\sigma_B^4(\kappa_B+2)$ in $\mathrm{Var}(U_a)$ from equation (12). It is

$$\sum_{ir}(ZZ^\mathsf{T})_{ir}(1-N_{i\bullet}^{-1}-N_{r\bullet}^{-1}+N_{i\bullet}^{-1}N_{r\bullet}^{-1})=\sum_j N_{\bullet j}^2-2\sum_{ij}Z_{ij}N_{i\bullet}^{-1}N_{\bullet j}+\sum_{ij}Z_{ij}N_{i\bullet}^{-1}N_{r\bullet}^{-1}$$

$$= \sum_j N_{\bullet j}^2 (1 + O(\epsilon)).$$

The third, fourth and fifth terms in $\mathrm{Var}(U_a)$ are all $O(\epsilon)$. The second term contains

$$\begin{aligned}\sum_{ir} N_{i\bullet}^{-1} N_{r\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir}((ZZ^{\mathsf{T}})_{ir} - 1) &\leqslant \sum_{ir} N_{i\bullet}^{-1} (ZZ^{\mathsf{T}})_{ir} \\ &= \sum_{irj} N_{i\bullet}^{-1} Z_{ij} Z_{rj} \\ &= \sum_{ij} Z_{ij} N_{i\bullet}^{-1} N_{\bullet j} \\ &= O(\epsilon).\end{aligned}$$

It follows that $\mathrm{Var}(U_a) = \sigma_B^4(\kappa_B + 2) \sum_j N_{\bullet j}^2 (1 + O(\epsilon))$. Similarly $\mathrm{Var}(U_b) = \sigma_A^4(\kappa_A + 2) \sum_i N_{i\bullet}^2 (1 + O(\epsilon))$.

The expression for $\mathrm{Var}(U_e)$ contains terms $\sigma_A^4(\kappa_A + 2)N^2 \sum_j N_{\bullet j}^2 + \sigma_B^4(\kappa_B + 2)N^2 \sum_i N_{i\bullet}^2$. All other terms are $O(\epsilon)$ times these two, mostly through $N \ll \sum_i N_{i\bullet}^2, \sum_j N_{\bullet j}^2 \ll N^2$. The coefficient of $\sigma_A^2 \sigma_B^2$ contains

$$N \sum_{ij} Z_{ij} N_{i\bullet} N_{\bullet j} \leqslant \epsilon N^2 \sum_{ij} Z_{ij} N_{i\bullet} = \epsilon N^2 \sum_i N_{i\bullet}^2$$

so it is of smaller order than the lead term, as well as

$$\sum_i N_{i\bullet}^2 \sum_j N_{\bullet j}^2 \leqslant \epsilon N^2 \sum_i N_{i\bullet}^2.$$

As a result

$$\mathrm{Var}(U_e) = \Bigl(\sigma_A^4(\kappa_A + 2)N^2 \sum_j N_{\bullet j}^2 + \sigma_B^4(\kappa_B + 2)N^2 \sum_i N_{i\bullet}^2\Bigr)(1 + O(\epsilon)).$$

Turning to the covariances

$$\begin{aligned}\mathrm{Cov}(U_a, U_b) &= \sigma_E^4(\kappa_E + 2) \sum_{ij} Z_{ij}(1 - N_{i\bullet}^{-1} - N_{\bullet j}^{-1} + N_{i\bullet}^{-1} N_{\bullet j}^{-1}) \\ &= \sigma_E^4(\kappa_E + 2)(N - R - C + O(R)) \\ &= \sigma_E^4(\kappa_E + 2)N(1 + O(\epsilon)).\end{aligned}$$

Next $\mathrm{Cov}(U_a, U_e)$ contains the term $\sigma_B^4(\kappa_B + 2)N \sum_{ij} Z_{ij} N_{\bullet j} = \sigma_B^4(\kappa_B + 2)N \sum_j N_{\bullet j}^2$. The terms appearing after that one are $O(N^2) = O(\epsilon N \sum_j N_{\bullet j}^2)$. The largest term preceding it is dominated by

$$\sum_i N_{i\bullet}^{-1} \Bigl(\sum_j Z_{ij} N_{\bullet j}\Bigr)^2 \leqslant \epsilon N \sum_i N_{i\bullet}^{-1} \Bigl(\sum_j Z_{ij} N_{\bullet j}\Bigr)\Bigl(\sum_j Z_{ij}\Bigr) = \epsilon N \sum_j N_{\bullet j}^2.$$

It follows that $\mathrm{Cov}(U_a, U_e) = \sigma_B^4(\kappa_B + 2)N \sum_j N_{\bullet j}^2 (1 + O(\epsilon))$ and similarly, $\mathrm{Cov}(U_b, U_e) = \sigma_A^4(\kappa_A + 2)N \sum_i N_{i\bullet}^2 (1 + O(\epsilon))$.

Next, using (19)

$$\begin{aligned}\mathrm{Var}(\hat{\sigma}_A^2) &= \Bigl(\frac{\mathrm{Var}(U_e)}{N^4} + \frac{\mathrm{Var}(U_a)}{N^2} - 2\frac{\mathrm{Cov}(U_a, U_e)}{N^3}\Bigr)(1 + O(\epsilon)) \\ &= \sigma_A^4(\kappa_A + 2)\frac{1}{N^2} \sum_i N_{i\bullet}^2 (1 + O(\epsilon)), \quad \text{and similarly} \\ \mathrm{Var}(\hat{\sigma}_B^2) &= \sigma_B^4(\kappa_B + 2)\frac{1}{N^2} \sum_j N_{\bullet j}^2 (1 + O(\epsilon)).\end{aligned}$$

The last variance is

$$\mathrm{Var}(\hat{\sigma}_E^2) = \Bigl(\frac{\mathrm{Var}(U_a)}{N^2} + \frac{\mathrm{Var}(U_b)}{N^2} + \frac{\mathrm{Var}(U_e)}{N^4} - \frac{2}{N^3}\mathrm{Cov}(U_a, U_e) - \frac{2}{N^3}\mathrm{Cov}(U_b, U_e) + \frac{2}{N^2}\mathrm{Cov}(U_a, U_b)\Bigr)(1 + O(\epsilon))$$

$$= \sigma_E^4(\kappa_E+2)\frac{1}{N}(1+O(\epsilon)).$$

Next we verify that these variance estimates are asymptotically uncorrelated. Ignoring the $1 + O(\epsilon)$ factors we have

$$\begin{aligned}
\mathrm{Cov}(\hat\sigma_A^2,\hat\sigma_B^2) &\doteq \frac{1}{N^4}\mathrm{Var}(U_e) - \frac{1}{N^3}\mathrm{Cov}(U_b,U_e) - \frac{1}{N^3}\mathrm{Cov}(U_a,U_e) + \frac{1}{N^2}\mathrm{Cov}(U_a,U_b)\\
&\doteq \frac{1}{N^2}\Big(\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 + \sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2)\Big)\\
&\quad - \frac{1}{N^2}\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 - \frac{1}{N^2}\sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2 + \frac{1}{N}\sigma_E^4(\kappa_E+2)\\
&= \frac{1}{N}\sigma_E^4(\kappa_E+2)
\end{aligned}$$

which is $O(\epsilon)$ times $\mathrm{Var}(\hat\sigma_A^2)$ and $\mathrm{Var}(\hat\sigma_B^2)$. Likewise

$$\begin{aligned}
\mathrm{Cov}(\hat\sigma_A^2,\hat\sigma_E^2) &\doteq \frac{1}{N^3}\mathrm{Cov}(U_a,U_e) + \frac{1}{N^3}\mathrm{Cov}(U_b,U_e) - \frac{1}{N^4}\mathrm{Var}(U_e) - \frac{1}{N^2}\mathrm{Var}(U_a) - \frac{1}{N^2}\mathrm{Cov}(U_a,U_b) + \frac{1}{N^3}\mathrm{Cov}(U_a,U_e)\\
&\doteq \sigma_B^4(\kappa_B+2)\frac{2}{N^2}\sum_j N_{\bullet j}^2 + \sigma_A^4(\kappa_A+2)\frac{1}{N^2}\sum_i N_{i\bullet}^2 - \Big(\sigma_A^4(\kappa_A+2)\sum_i N_{i\bullet}^2 + \sigma_B^4(\kappa_B+2)\sum_j N_{\bullet j}^2\Big)\frac{1}{N^2}\\
&\quad - \sigma_B^4(\kappa_B+2)\frac{1}{N^2}\sum_j N_{\bullet j}^2 - \sigma_E^4(\kappa_E+2)\frac{1}{N}\\
&= -\sigma_E^4(\kappa_E+2)\frac{1}{N}
\end{aligned}$$

which is much smaller than $\mathrm{Var}(\hat\sigma_A^2)$. Similarly $\mathrm{Cov}(\hat\sigma_B^2,\hat\sigma_E^2) \doteq -\sigma_E^4(\kappa_E+2)/N$, is much smaller than $\mathrm{Var}(\hat\sigma_B^2)$.

17 Estimating Kurtoses

To estimate the kurtoses κ_A, κ_B and κ_E in the above variance expressions, it suffices to estimate fourth central moments such as $\mu_{A,4} = \sigma_A^4(\kappa_A+3)$ and similarly defined $\mu_{B,4}$ and $\mu_{E,4}$. Given $\hat\sigma_A^2$, $\hat\sigma_B^2$, and $\hat\sigma_E^2$, we can do this via GMM. Consider the following estimating equations and their expectations,

$$\begin{aligned}
W_a &= \frac{1}{2}\sum_{ijj'}\frac{1}{N_{i\bullet}}Z_{ij}Z_{ij'}(Y_{ij}-Y_{ij'})^4\\
W_b &= \frac{1}{2}\sum_{ii'j}\frac{1}{N_{\bullet j}}Z_{ij}Z_{i'j}(Y_{ij}-Y_{i'j})^4\\
W_e &= \frac{1}{2}\sum_{ii'jj'}Z_{ij}Z_{i'j'}(Y_{ij}-Y_{i'j'})^4
\end{aligned}$$

Using previous results,

$$\begin{aligned}
\mathbb{E}(W_a) &= \frac{1}{2}\sum_{ijj'}\frac{1}{N_{i\bullet}}Z_{ij}Z_{ij'}\mathbb{E}[(Y_{ij}-Y_{ij'})^4\big)\\
&= \frac{1}{2}\sum_{ijj'}\frac{1}{N_{i\bullet}}Z_{ij}Z_{ij'}\mathbb{E}\big((b_j-b_{j'}+e_{ij}-e_{ij'})^4\big)\\
&= \frac{1}{2}\sum_{ijj'}\frac{1}{N_{i\bullet}}Z_{ij}Z_{ij'}\mathbb{E}\big((b_j-b_{j'})^4+6(b_j-b_{j'})^2(e_{ij}-e_{ij'})^2+(e_{ij}-e_{ij'})^4\big)\\
&= \frac{1}{2}\sum_{ijj'}\frac{1}{N_{i\bullet}}Z_{ij}Z_{ij'}\big(2\mu_{B,4}+6\sigma_B^4+24\sigma_B^2\sigma_E^2+2\mu_{E,4}+6\sigma_E^4\big)(1-1_{j=j'})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{ijj'} \frac{1}{N_{i\bullet}} Z_{ij} Z_{ij'} \big(\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4\big)(1 - 1_{j=j'}) \\
&= \sum_{i} \big((\kappa_B + 2)\sigma_B^4 + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2 + (\kappa_E + 2)\sigma_E^4 + 3\sigma_E^4\big)(N_{i\bullet} - 1) \\
&= (N - R)\big(\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4\big).
\end{aligned}
$$

By symmetry,

$$
\mathbb{E}(W_b) = (N - C)\big(\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2 + \mu_{E,4} + 3\sigma_E^4\big).
$$

Next

$$
\begin{aligned}
\mathbb{E}(W_e) &= \frac{1}{2} \sum_{ii'jj'} Z_{ij} Z_{i'j'} \mathbb{E}\big((Y_{ij} - Y_{i'j'})^4\big) \\
&= \frac{1}{2} \sum_{ii'jj'} Z_{ij} Z_{i'j'} \mathbb{E}\big((a_i - a_{i'} + b_j - b_{j'} + e_{ij} - e_{i'j'})^4\big) \\
&= \frac{1}{2} \sum_{ii'jj'} Z_{ij} Z_{i'j'} \mathbb{E}\big((a_i - a_{i'})^4 + 6(a_i - a_{i'})^2(b_j - b_{j'})^2 + 6(a_i - a_{i'})^2(e_{ij} - e_{i'j'})^2 + (b_j - b_{j'})^4 \\
&\qquad\qquad + 6(b_j - b_{j'})^2(e_{ij} - e_{i'j'})^2 + (e_{ij} - e_{i'j'})^4\big) \\
&= \frac{1}{2} \sum_{ii'jj'} Z_{ij} Z_{i'j'} \big((2\mu_{A,4} + 6\sigma_A^4)(1 - 1_{i=i'}) + 24\sigma_A^2\sigma_B^2(1 - 1_{i=i'})(1 - 1_{j=j'}) + 24\sigma_A^2\sigma_E^2(1 - 1_{i=i'}) \\
&\qquad\qquad + (2\mu_{B,4} + 6\sigma_B^4)(1 - 1_{j=j'}) + 24\sigma_B^2\sigma_E^2(1 - 1_{j=j'}) + (2\mu_{E,4} + 6\sigma_E^4)(1 - 1_{i=i'}1_{j=j'})\big) \\
&= (\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2) \sum_{ii'jj'} Z_{ij} Z_{i'j'}(1 - 1_{i=i'}) + (\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2) \sum_{ii'jj'} Z_{ij} Z_{i'j'}(1 - 1_{j=j'}) \\
&\quad + (\mu_{E,4} + 3\sigma_E^4) \sum_{ii'jj'} Z_{ij} Z_{i'j'}(1 - 1_{i=i'}1_{j=j'}) + 12\sigma_A^2\sigma_B^2 \sum_{ii'jj'} Z_{ij} Z_{i'j'}(1 - 1_{i=i'} - 1_{j=j'} + 1_{i=i'}1_{j=j'}) \\
&= (\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2)(N^2 - \sum_{ijj'} Z_{ij} Z_{ij'}) + (\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2)(N^2 - \sum_{ii'j} Z_{ij} Z_{i'j}) \\
&\quad + (\mu_{E,4} + 3\sigma_E^4)N(N - 1) + 12\sigma_A^2\sigma_B^2(N^2 - \sum_{ijj'} Z_{ij} Z_{ij'} - \sum_{ii'j} Z_{ij} Z_{i'j} + N) \\
&= (\mu_{A,4} + 3\sigma_A^4 + 12\sigma_A^2\sigma_E^2)(N^2 - \sum_{i} N_{i\bullet}^2) + (\mu_{B,4} + 3\sigma_B^4 + 12\sigma_B^2\sigma_E^2)(N^2 - \sum_{j} N_{\bullet j}^2) \\
&\quad + (\mu_{E,4} + 3\sigma_E^4)N(N - 1) + 12\sigma_A^2\sigma_B^2(N^2 - \sum_{i} N_{i\bullet}^2 - \sum_{j} N_{\bullet j}^2 + N).
\end{aligned}
$$

These expectations are all linear in the fourth moments. Therefore, given estimates of σ_A^2, σ_B^2, and σ_E^2, we can solve another three-by-three system of equations to get estimates of the fourth moments.

Letting M be the matrix in equation (106) we find that

$$
\begin{pmatrix} \mathbb{E}(W_a) \\ \mathbb{E}(W_b) \\ \mathbb{E}(W_e) \end{pmatrix} = M \begin{pmatrix} \mu_{A,4} \\ \mu_{B,4} \\ \mu_{E,4} \end{pmatrix} + \begin{pmatrix} 3(N - R)\sigma_B^4 + 12(N - R)\sigma_B^2\sigma_E^2 + 3(N - R)\sigma_E^4 \\ 3(N - C)\sigma_A^4 + 12(N - C)\sigma_A^2\sigma_E^2 + 3(N - C)\sigma_E^4 \\ H \end{pmatrix}
$$

where

$$
\begin{aligned}
H = (3\sigma_A^4 + 12\sigma_A^2\sigma_E^2)(N^2 - \sum_{i} N_{i\bullet}^2) &+ (3\sigma_B^4 + 12\sigma_B^2\sigma_E^2)(N^2 - \sum_{j} N_{\bullet j}^2) \\
&+ 3\sigma_E^4 N(N - 1) + 12\sigma_A^2\sigma_B^2(N^2 - \sum_{i} N_{i\bullet}^2 - \sum_{j} N_{\bullet j}^2 + N).
\end{aligned}
$$

For plug-in method of moment estimators we replace expected W-statistics by their sample quantities, replace the variance components by their estimates and solve the matrix equation getting $\hat{\mu}_{A,4}$ et cetera. Then $\hat{\kappa}_A = \hat{\mu}_{A,4}/\hat{\sigma}_A^4 - 3$ and so on.

18 Best linear predictor

Here we predict consider linear predicton of Y_{ij}. We begin with predictions of the form $\hat{Y}_{ij} = \hat{Y}_{ij}(\lambda) = \sum_{rs} \lambda_{rs} Z_{rs} Y_{rs}$. Then we consider predictions of a reduced form that consider only the totals in row i, in row j and in the whole data set.

18.1 Proof of Lemma 5.1

Let $\hat{Y}_{ij} = \sum_{rs} Z_{ij}\lambda_{ij}Y_{ij}$ and $L = \mathbb{E}((Y_{ij} - \hat{Y}_{ij})^2)$. Then

$$L = \mu^2\Bigl(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Bigr)^2 + \mathrm{Var}(Y_{ij}) + \mathrm{Var}(\hat{Y}_{ij}) - 2\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij}).$$

First $\mathrm{Var}(Y_{ij}) = \sigma_A^2 + \sigma_B^2 + \sigma_E^2$. Next

$$\begin{aligned}\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij}) &= \sum_{rs} \lambda_{rs} Z_{rs}\bigl(\sigma_A^2 1_{i=r} + \sigma_B^2 1_{j=s} + \sigma_E^2 1_{i=r} 1_{j=s}\bigr)\\ &= \sigma_A^2 \sum_s \lambda_{is} Z_{is} + \sigma_B^2 \sum_r \lambda_{rj} Z_{rj} + \sigma_E^2 \lambda_{ij}^2 Z_{ij},\end{aligned}$$

and finally

$$\begin{aligned}\mathrm{Var}(\hat{Y}_{ij}) &= \sum_{rs}\sum_{r's'} \lambda_{rs}\lambda_{r's'} Z_{rs} Z_{r's'}\bigl(\sigma_A^2 1_{r=r'} + \sigma_B^2 1_{s=s'} + \sigma_E^2 1_{r=r'} 1_{s=s'}\bigr)\\ &= \sigma_A^2 \sum_{rss'} \lambda_{rs}\lambda_{rs'} Z_{rs} Z_{rs'} + \sigma_B^2 \sum_{rsr'} \lambda_{rs}\lambda_{r's} Z_{rs} Z_{r's} + \sigma_E^2 \sum_{rs} \lambda_{rs}^2 Z_{rs}.\end{aligned}$$

Thus

$$\begin{aligned}L = \mu^2\Bigl(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Bigr)^2 &+ \sigma_A^2 + \sigma_B^2 + \sigma_E^2\\ &+ \sigma_A^2 \sum_{rss'} \lambda_{rs}\lambda_{rs'} Z_{rs} Z_{rs'} + \sigma_B^2 \sum_{rsr'} \lambda_{rs}\lambda_{r's} Z_{rs} Z_{r's} + \sigma_E^2 \sum_{rs} \lambda_{rs}^2 Z_{rs}\\ &- 2\Bigl(\sigma_A^2 \sum_s \lambda_{is} Z_{is} + \sigma_B^2 \sum_r \lambda_{rj} Z_{rj} + \sigma_E^2 \lambda_{ij}^2 Z_{ij}\Bigr).\end{aligned}$$

Now suppose that we consider the loss $\widetilde{L} = \mathbb{E}(((\mu + a_i + b_j) - \hat{Y}_{ij})^2)$. To do so we replace $\mathrm{Var}(\hat{Y}_{ij})$ and $\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij})$ above by $\mathrm{Var}(a_i + b_j)$ and $\mathrm{Cov}(a_i + b_j, \hat{Y}_{ij})$ respectively, yielding

$$\begin{aligned}\widetilde{L} = \mu^2\Bigl(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Bigr)^2 &+ \sigma_A^2 + \sigma_B^2\\ &+ \sigma_A^2 \sum_{rss'} \lambda_{rs}\lambda_{rs'} Z_{rs} Z_{rs'} + \sigma_B^2 \sum_{rsr'} \lambda_{rs}\lambda_{r's} Z_{rs} Z_{r's} + \sigma_E^2 \sum_{rs} \lambda_{rs}^2 Z_{rs}\\ &- 2\Bigl(\sigma_A^2 \sum_s \lambda_{is} Z_{is} + \sigma_B^2 \sum_r \lambda_{rj} Z_{rj}\Bigr).\end{aligned}$$

18.2 Stationary conditions

The partial derivative of $\widetilde{L}$ with respect to $\lambda_{r''s''}$ is

$$\begin{aligned}2\mu^2\Bigl(1 - \sum_{rs} \lambda_{rs} Z_{rs}\Bigr)(-Z_{r''s''}) &+ 2\sigma_E^2 \lambda_{r''s''} Z_{r''s''}\\ &+ \sigma_A^2 \sum_{rss'} Z_{rs} Z_{rs'}(\lambda_{rs'} 1_{rs=r''s''} + \lambda_{rs} 1_{rs'=r''s''})\end{aligned}$$

$$
\begin{aligned}
&+ \sigma_B^2 \sum_{rsr'} Z_{rs} Z_{r's} (\lambda_{r's} 1_{rs=r''s''} + \lambda_{rs} 1_{r's=r''s''}) \\
&- 2\sigma_A^2 \sum_s Z_{is} 1_{is=r''s''} - 2\sigma_B^2 \sum_r Z_{rj} 1_{rj=r''s''}.
\end{aligned}
$$

After taking account of the indicator functions we get

$$
\begin{aligned}
&2Z_{r''s''}\Biggl(\mu^2\Bigl(1-\sum_{rs}\lambda_{rs}Z_{rs}\Bigr)(-1) + \sigma_E^2\lambda_{r''s''} + \sigma_A^2\sum_{s'} Z_{r''s'}\lambda_{r''s'} + \sigma_B^2 \sum_{r'} Z_{r's''}\lambda_{r's''} \\
&- \sigma_A^2 Z_{is''} 1_{i=r''} - \sigma_B^2 Z_{r''j} 1_{j=s''}\Biggr).
\end{aligned}
$$

We can replace $Z_{is''}1_{i=r''}$ by $1_{i=r''}$ because of the leading factor $Z_{r''s''}$. This and a corresponding change to the coefficient of σ_B^2 yield

$$
2Z_{r''s''}\Biggl(\mu^2\Bigl(1-\sum_{rs}\lambda_{rs}Z_{rs}\Bigr)(-1) + \sigma_E^2\lambda_{r''s''} + \sigma_A^2\sum_{s'} Z_{r''s'}\lambda_{r''s'} + \sigma_B^2 \sum_{r'} Z_{r's''}\lambda_{r's''} - \sigma_A^2 1_{i=r''} - \sigma_B^2 1_{j=s''}\Biggr).
$$

The simplified expression no longer requires the double primes and so we find that the partial derivative of $\widetilde{L}$ with respect to λ_{rs} is

$$
2Z_{rs}\Biggl(\mu^2\Bigl(\sum_{r's'}\lambda_{r's'}Z_{r's'} - 1\Bigr) + \sigma_E^2\lambda_{rs} + \sigma_A^2\sum_{s'} Z_{rs'}\lambda_{rs'} + \sigma_B^2 \sum_{r'} Z_{r's}\lambda_{r's} - \sigma_A^2 1_{i=r} - \sigma_B^2 1_{j=s}\Biggr).
$$

18.3 Proof of Lemma 5.2

Here we consider

$$
\hat{Y}_{ij} = \lambda_0 Y_{\bullet\bullet} + \lambda_a Y_{i\bullet} + \lambda_b Y_{\bullet j}
$$

where

$$
Y_{\bullet\bullet} = \sum_{rs} Z_{rs} Y_{rs}, \quad Y_{i\bullet} = \sum_s Z_{is} Y_{is}, \quad \text{and} \quad Y_{\bullet j} = \sum_r Z_{rj} Y_{rj}.
$$

The mean squared error is $L = \mathbb{E}((Y_{ij} - \hat{Y}_{ij})^2)$. Expanding it we get

$$
\begin{aligned}
L = {}& \mu^2\bigl(1 - (\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j})\bigr)^2 + \mathrm{Var}(Y_{ij}) + \lambda_0^2 \mathrm{Var}(Y_{\bullet\bullet}) + \lambda_a^2 \mathrm{Var}(Y_{i\bullet}) + \lambda_b^2 \mathrm{Var}(Y_{\bullet j}) \\
&- 2\lambda_0 \mathrm{Cov}(Y_{ij}, Y_{\bullet\bullet}) - 2\lambda_a \mathrm{Cov}(Y_{ij}, Y_{i\bullet}) - 2\lambda_b \mathrm{Cov}(Y_{ij}, Y_{\bullet j}) \\
&+ 2\lambda_0\lambda_a \mathrm{Cov}(Y_{\bullet\bullet}, Y_{i\bullet}) + 2\lambda_0\lambda_b \mathrm{Cov}(Y_{\bullet\bullet}, Y_{\bullet j}) + 2\lambda_a\lambda_b \mathrm{Cov}(Y_{i\bullet}, Y_{\bullet j}).
\end{aligned}
$$

As before $\mathrm{Var}(Y_{ij}) = \sigma_A^2 + \sigma_B^2 + \sigma_E^2$. We set about finding the other terms.

First

$$
\begin{aligned}
\mathrm{Var}(Y_{\bullet\bullet}) &= \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N, \\
\mathrm{Var}(Y_{i\bullet}) &= \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}, \quad \text{and} \\
\mathrm{Var}(Y_{\bullet j}) &= \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.
\end{aligned}
$$

Second

$$
\begin{aligned}
\mathrm{Cov}(Y_{ij}, Y_{\bullet\bullet}) &= \sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}, \\
\mathrm{Cov}(Y_{ij}, Y_{i\bullet}) &= \sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}, \quad \text{and} \\
\mathrm{Cov}(Y_{ij}, Y_{\bullet j}) &= \sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}.
\end{aligned}
$$

The remaining terms use somewhat longer arguments.

$$
\mathrm{Cov}(Y_{i\bullet}, Y_{\bullet\bullet}) = \sum_{rss'} Z_{rs} Z_{is'} \mathrm{Cov}(Y_{rs}, Y_{is'})
$$

$$= \sum_{rss'} Z_{rs} Z_{is'} \big(1_{i=r}\sigma_A^2 + 1_{s=s'}\sigma_B^2 + 1_{i=r}1_{s=s'}\sigma_E^2\big)$$
$$= \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}, \quad \text{and then}$$
$$\mathrm{Cov}(Y_{\bullet j}, Y_{\bullet\bullet}) = \sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}$$

by symmetry. Finally

$$\begin{aligned}
\mathrm{Cov}(Y_{i\bullet}, Y_{\bullet j}) &= \sum_{rs} Z_{is} Z_{rj} \mathrm{Cov}(Y_{is}, Y_{rj}) \\
&= \sum_{rs} Z_{is} Z_{rj} \big(\sigma_A^2 1_{i=r} + \sigma_B^2 1_{j=s} + \sigma_E^2 1_{i=r} 1_{j=s}\big) \\
&= \sigma_A^2 \sum_s Z_{is} Z_{ij} + \sigma_B^2 \sum_r Z_{ij} Z_{rj} + \sigma_E^2 Z_{ij} \\
&= Z_{ij} \big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big).
\end{aligned}$$

Combining these pieces we find that

$$\begin{aligned}
L = {}& \mu^2\big(1 - \lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\big)^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2 + \lambda_0^2\Big(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big) \\
& + \lambda_a^2\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big) + \lambda_b^2\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) \\
& - 2\lambda_0\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big) - 2\lambda_a\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}\Big) - 2\lambda_b\Big(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\Big) \\
& + 2\lambda_0\lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big) + 2\lambda_0\lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) \\
& + 2\lambda_a\lambda_b Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big).
\end{aligned}$$

Now suppose we consider instead $\widetilde{L} = \mathbb{E}((\mu + a_i + b_j - \hat{Y}_{ij})^2)$. Then we must replace $\mathrm{Var}(\hat{Y}_{ij})$ by $\mathrm{Var}(\mu + a_i + b_j) = \sigma_A^2 + \sigma_B^2$ and remove the $\sigma_E^2 Z_{ij}$ terms from the covariances with Y_{ij}. The result is

$$\begin{aligned}
\widetilde{L} = {}& \mu^2\big(1 - \lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\big)^2 + \sigma_A^2 + \sigma_B^2 + \lambda_0^2\Big(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big) \\
& + \lambda_a^2\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big) + \lambda_b^2\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) \\
& - 2\lambda_0\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Big) - 2\lambda_a\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij}\Big) - 2\lambda_b\Big(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j}\Big) \\
& + 2\lambda_0\lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big) + 2\lambda_0\lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big) \\
& + 2\lambda_a\lambda_b Z_{ij}\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big).
\end{aligned}$$

18.4 Proof of Theorem 5.1

From the result of Lemma 5.2, we see that $\widetilde{L}$ is quadratic in λ. Since $\widetilde{L}$ is bounded below by 0, it follows that $\widetilde{L}$ attains its minimum on $\mathbb{R}^3$, which would be any solution of the stationarity condition $\nabla_\lambda \widetilde{L} = 0$. We find the components of this gradient.

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial \lambda_0}\widetilde{L} = {}& N\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) + \lambda_0\Big(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big) - \Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Big) \\
& + \lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big) + \lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)
\end{aligned}$$

$$\frac{1}{2}\frac{\partial}{\partial\lambda_a}\widetilde{L} = N_{i\bullet}\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) + \lambda_a\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Bigr) - \Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij}\Bigr)$$
$$+ \lambda_0\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2\sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Bigr) + \lambda_b Z_{ij}\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Bigr), \quad \text{and}$$
$$\frac{1}{2}\frac{\partial}{\partial\lambda_b}\widetilde{L} = N_{\bullet j}\mu^2(\lambda_0 N + \lambda_a N_{i\bullet} + \lambda_b N_{\bullet j} - 1) + \lambda_b\Bigl(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr) - \Bigl(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j}\Bigr)$$
$$+ \lambda_0\Bigl(\sigma_A^2\sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr) + \lambda_a Z_{ij}\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Bigr).$$

We write this as

$$H\lambda^* = c$$

where

$$c = \begin{pmatrix} N\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} \\ N_{i\bullet}\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} \\ N_{\bullet j}\mu^2 + \sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j}\end{pmatrix} = \begin{pmatrix} N & N_{i\bullet} & N_{\bullet j} \\ N_{i\bullet} & N_{i\bullet} & Z_{ij} \\ N_{\bullet j} & Z_{ij} & N_{\bullet j}\end{pmatrix}\begin{pmatrix}\mu^2 \\ \sigma_A^2 \\ \sigma_B^2\end{pmatrix}$$

and H is a symmetric matrix with upper triangle

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ * & H_{22} & H_{23} \\ * & * & H_{33}\end{pmatrix}$$

with elements

$$H_{11} = \mu^2 N^2 + \sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N$$
$$H_{12} = \mu^2 N N_{i\bullet} + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2\sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}$$
$$H_{13} = \mu^2 N N_{\bullet j} + \sigma_A^2\sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}$$
$$H_{22} = \mu^2 N_{i\bullet}^2 + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}$$
$$H_{23} = \mu^2 N_{i\bullet}N_{\bullet j} + \sigma_A^2 Z_{ij}N_{i\bullet} + \sigma_B^2 Z_{ij}N_{\bullet j} + \sigma_E^2 Z_{ij}, \quad \text{and}$$
$$H_{33} = \mu^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.$$

Using $T_{i\bullet} \equiv \sum_s Z_{is}N_{\bullet s}$ and $T_{\bullet j} \equiv \sum_r Z_{rj}N_{r\bullet}$ some of these simplify:

$$H_{12} = \mu^2 N N_{i\bullet} + \sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 T_{i\bullet} + \sigma_E^2 N_{i\bullet}, \quad \text{and}$$
$$H_{13} = \mu^2 N N_{\bullet j} + \sigma_A^2 T_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}.$$

18.5 Proof of Theorem 5.2

To begin with, we note that $N_{\bullet j} = \sum_r Z_{rj} \leqslant \sum_r N_{r\bullet}Z_{rj} \leqslant \epsilon N$. We write

$$\begin{pmatrix}\lambda_0^* \\ \lambda_b^*\end{pmatrix} = \frac{1}{\det\tilde{H}}\begin{pmatrix} H_{33} & -H_{13} \\ -H_{31} & H_{11}\end{pmatrix}\begin{pmatrix} c_1 \\ c_3\end{pmatrix}.$$

Then

$$\begin{aligned}\det\tilde{H}\lambda_0^* &= H_{33}c_1 - H_{13}c_3 \\ &= N_{\bullet j}(\mu^2 N_{\bullet j} + \sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2)(N\mu^2 + N_{\bullet j}\sigma_B^2) \\ &\quad - (\mu^2 N N_{\bullet j} + \sigma_A^2\sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j})N_{\bullet j}(\mu^2 + \sigma_B^2) \\ &= \mu^2\Bigl(\mu^2 N N_{\bullet j}^2 + \sigma_A^2 N N_{\bullet j} + \sigma_B^2 N N_{\bullet j}^2 + \sigma_E^2 N N_{\bullet j} - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2\Bigr)\end{aligned}$$

$$
\begin{aligned}
&+ \sigma_B^2\Big(\mu^2 N_{\bullet j}^3 + \sigma_A^2 N_{\bullet j}^2 + \sigma_B^2 N_{\bullet j}^3 + \sigma_E^2 N_{\bullet j}^2 - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2\Big)\\
&= \mu^2\Big(\sigma_A^2 N N_{\bullet j} + \sigma_B^2 N N_{\bullet j}^2 + \sigma_E^2 N N_{\bullet j} - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2\Big)\\
&\quad + \sigma_B^2\Big(\mu^2 N_{\bullet j}^3 + \sigma_A^2 N_{\bullet j}^2 - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet}\Big)\\
&= \mu^2\Big(\sigma_A^2 N N_{\bullet j} + \sigma_E^2 N N_{\bullet j} - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_E^2 N_{\bullet j}^2\Big)\\
&\quad + \sigma_B^2\Big(\sigma_A^2 N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet}\Big)\\
&= \mu^2(\sigma_A^2 + \sigma_E^2) N N_{\bullet j}(1 + O(\epsilon)).
\end{aligned}
$$

and

$$
\begin{aligned}
\det \tilde{H}\lambda_b^* &= H_{11}c_3 - H_{31}c_1\\
&= (\mu^2 N^2 + \sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N)N_{\bullet j}(\mu^2 + \sigma_B^2)\\
&\quad - (\mu^2 N N_{\bullet j} + \sigma_A^2\sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j})(N\mu^2 + N_{\bullet j}\sigma_B^2)\\
&= \mu^2\Big(\mu^2 N^2 N_{\bullet j} + \sigma_A^2 N_{\bullet j}\sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j}\sum_s N_{\bullet s}^2 + \sigma_E^2 N N_{\bullet j}\\
&\quad - \mu^2 N^2 N_{\bullet j} - \sigma_A^2 N\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N N_{\bullet j}^2 - \sigma_E^2 N N_{\bullet j}\Big)\\
&\quad + \sigma_B^2\Big(\mu^2 N^2 N_{\bullet j} + \sigma_A^2 N_{\bullet j}\sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j}\sum_s N_{\bullet s}^2 + \sigma_E^2 N N_{\bullet j}\\
&\quad - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2\Big)\\
&= \mu^2\Big(\sigma_A^2 N_{\bullet j}\sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j}\sum_s N_{\bullet s}^2 - \sigma_A^2 N\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N N_{\bullet j}^2\Big)\\
&\quad + \sigma_B^2\Big(\mu^2 N^2 N_{\bullet j} + \sigma_A^2 N_{\bullet j}\sum_r N_{r\bullet}^2 + \sigma_B^2 N_{\bullet j}\sum_s N_{\bullet s}^2 + \sigma_E^2 N N_{\bullet j}\\
&\quad - \mu^2 N N_{\bullet j}^2 - \sigma_A^2 N_{\bullet j}\sum_r Z_{rj}N_{r\bullet} - \sigma_B^2 N_{\bullet j}^3 - \sigma_E^2 N_{\bullet j}^2\Big)\\
&= \mu^2\sigma_B^2 N^2 N_{\bullet j}(1 + O(\epsilon)).
\end{aligned}
$$

Thus

$$
\frac{\lambda_0^*}{\lambda_b^*} = \frac{\sigma_A^2 + \sigma_E^2}{\sigma_B^2 N}(1 + O(\epsilon)).
$$

Next

$$
\begin{aligned}
\det \tilde{H} &= H_{11}H_{33} - H_{13}^2\\
&= \Big(\mu^2 N^2 + \sigma_A^2\sum_r N_{r\bullet}^2 + \sigma_B^2\sum_s N_{\bullet s}^2 + \sigma_E^2 N\Big)\Big(\mu^2 N_{\bullet j}^2 + \sigma_B^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_E^2 N_{\bullet j}\Big)\\
&\quad - \Big(\mu^2 N N_{\bullet j} + \sigma_A^2\sum_r N_{r\bullet}Z_{rj} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)^2\\
&\approx \mu^2 N^2 N_{\bullet j}^2(\mu^2 + \sigma_B^2) - (\mu^2 N N_{\bullet j})^2\\
&= \mu^2 N^2 N_{\bullet j}^2\sigma_B^2.
\end{aligned}
$$

As a result the prediction for a new row in a large column is essentially that column average plus $O(1/N_{\bullet j})$ times the global average.

18.6 Special case $N_{i\bullet} = 0$ and $N_{\bullet j} = 1$

Now suppose that we have no data in the target row and exactly one older observation in the target column. Let i' be the single row with $Z_{i'j} = 1$. There are enough large rows and columns that the usual conditions $N \ll \sum_i N_{i\bullet}^2 \ll N^2$ hold but there are also some lightly observed rows and columns. Then

$$\tilde{c} = \begin{pmatrix} N & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu^2 \\ \sigma_A^2 \\ \sigma_B^2 \end{pmatrix} = \begin{pmatrix} N\mu^2 + \sigma_B^2 \\ \mu^2 + \sigma_B^2 \end{pmatrix},$$

and

$$\begin{aligned} H_{11} &= \mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N \\ H_{13} &= \mu^2 N N_{\bullet j} + \sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j} \\ &= \mu^2 N + \sigma_A^2 N_{i'\bullet} + \sigma_B^2 + \sigma_E^2, \quad \text{and} \\ H_{33} &= \mu^2 N_{\bullet j}^2 + \sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j} \\ &= \mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2. \end{aligned}$$

Then

$$\begin{pmatrix} \lambda_0^* \\ \lambda_b^* \end{pmatrix} = \frac{1}{H_{11}H_{33} - H_{13}^2} \begin{pmatrix} H_{33} & -H_{13} \\ -H_{13} & H_{11} \end{pmatrix} \begin{pmatrix} N\mu^2 + \sigma_B^2 \\ \mu^2 + \sigma_B^2 \end{pmatrix}.$$

The determinant is

$$\begin{aligned} &\Bigl(\mu^2 N^2 + \sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Bigr)\Bigl(\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2\Bigr) - \Bigl(\mu^2 N + \sigma_A^2 N_{i'\bullet} + \sigma_B^2 + \sigma_E^2\Bigr)^2 \\ &\approx \mu^2 N^2 \Bigl(\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2\Bigr) - \mu^4 N^2 \\ &= \mu^2 N^2 (\sigma_A^2 + \sigma_B^2 + \sigma_E^2). \end{aligned}$$

The numerator for λ_0^* is

$$\begin{aligned} &H_{33}(N\mu^2 + \sigma_B^2) - H_{13}(\mu^2 + \sigma_B^2) \\ &\approx (\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2) N\mu^2 - (\mu^2 N)(\mu^2 + \sigma_B^2) \\ &= (\sigma_A^2 + \sigma_E^2) N\mu^2, \end{aligned}$$

and so

$$\lambda_0^* \approx \frac{1}{N} \frac{\sigma_A^2 + \sigma_E^2}{\sigma_A^2 + \sigma_B^2 + \sigma_E^2}.$$

Similarly, the numerator for λ_b^* is

$$\begin{aligned} &-H_{13}(N\mu^2 + \sigma_B^2) + H_{11}(\mu^2 + \sigma_B^2) \\ &\approx -(\mu^2 N) N\mu^2 + \mu^2 N^2 (\mu^2 + \sigma_B^2) \\ &= \mu^2 \sigma_B^2 N^2 \end{aligned}$$

and so

$$\lambda_b^* \approx \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2 + \sigma_E^2}.$$

In this case, the prediction for Y_{ij} is

$$\frac{\sigma_B^2 \overline{Y}_{\bullet j} + (\sigma_A^2 + \sigma_E^2)\overline{Y}_{\bullet\bullet}}{\sigma_A^2 + \sigma_B^2 + \sigma_E^2}.$$

19 Asymptotic weights: proof of Theorem 5.3

Here we have

$$
\begin{aligned}
&1 \leqslant N_{i\bullet} \leqslant \epsilon N, && 1 \leqslant N_{\bullet j} \leqslant \epsilon N, && N_{i\bullet} \leqslant \epsilon N_{i\bullet}^2,\\
&N_{\bullet j} \leqslant \epsilon N_{\bullet j}^2, && N \leqslant \epsilon N^2, && \sum_r N_{r\bullet}^2 \leqslant \epsilon N^2,\\
&\sum_s N_{\bullet s}^2 \leqslant \epsilon N^2, && \sum_r N_{r\bullet} Z_{rj} \leqslant \epsilon N N_{\bullet j}, \quad \text{and} && \sum_s N_{\bullet s} Z_{is} \leqslant \epsilon N N_{i\bullet}.
\end{aligned}
$$

The first five follow easily from $1 < 1/\epsilon \leqslant N_{i\bullet}, N_{\bullet j} \leqslant \epsilon N$. The last four follow from the others. For instance $\sum_r N_{r\bullet}^2 \leqslant \sum_r N_{r\bullet}(\epsilon N) = \epsilon N^2$, and $\sum_r N_{r\bullet} Z_{rj} \leqslant \sum_r Z_{rj}(\epsilon N) = \epsilon N_{\bullet j} N$. We also have $0 < m \leqslant \mu^2, \sigma_A^2, \sigma_B^2, \sigma_E^2 \leqslant M < \infty$.

Then

$$
H = \begin{pmatrix} \mu^2 N^2 & \mu^2 N N_{i\bullet} & \mu^2 N N_{\bullet j} \\ \mu^2 N N_{i\bullet} & (\mu^2 + \sigma_A^2) N_{i\bullet}^2 & \mu^2 N_{i\bullet} N_{\bullet j} \\ \mu^2 N N_{\bullet j} & \mu^2 N_{i\bullet} N_{\bullet j} & (\mu^2 + \sigma_B^2) N_{\bullet j}^2 \end{pmatrix} (1 + O(\epsilon))
$$

and using symbolic computation (via Wolfram|Alpha, September 6, 2015)

$$
H^{-1} = \begin{pmatrix} \dfrac{\mu^2(\sigma_A^2 + \sigma_B^2) + \sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2\mu^2 N^2} & \dfrac{-1}{\sigma_A^2 N_{i\bullet} N} & \dfrac{-1}{\sigma_B^2 N_{\bullet j} N} \\ \dfrac{-1}{\sigma_A^2 N_{i\bullet} N} & \dfrac{1}{\sigma_A^2 N_{i\bullet}^2} & 0 \\ \dfrac{-1}{\sigma_B^2 N_{\bullet j} N} & 0 & \dfrac{1}{\sigma_B^2 N_{\bullet j}^2} \end{pmatrix} (1 + O(\epsilon)).
$$

The determinant of H^{-1} is $(\sigma_A^2\sigma_B^2\mu^2 N_{i\bullet}^2 N_{\bullet j}^2 N^2)^{-1}(1 + O(\epsilon))$, so we need $N_{i\bullet} \geqslant 1$ and $N_{\bullet j} \geqslant 1$ to make matrix inversion a continuous operation. Similarly

$$
c = \begin{pmatrix} N\mu^2 \\ N_{i\bullet}(\mu^2 + \sigma_A^2) \\ N_{\bullet j}(\mu^2 + \sigma_B^2) \end{pmatrix} (1 + O(\epsilon)).
$$

Thus ignoring the $O(\epsilon)$ terms

$$
\begin{aligned}
\lambda_0^* &\doteq \Bigl(\frac{\mu^2(\sigma_A^2 + \sigma_B^2) + \sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2\mu^2 N^2}\Bigr) N\mu^2 - \Bigl(\frac{1}{\sigma_A^2 N_{i\bullet} N}\Bigr) N_{i\bullet}(\mu^2 + \sigma_A^2) - \Bigl(\frac{1}{\sigma_B^2 N_{\bullet j} N}\Bigr) N_{\bullet j}(\mu^2 + \sigma_B^2)\\
&= \frac{\mu^2(\sigma_A^2 + \sigma_B^2) + \sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2 + \sigma_A^2}{\sigma_A^2 N} - \frac{\mu^2 + \sigma_B^2}{\sigma_B^2 N}\\
&= \frac{\mu^2(\sigma_A^2 + \sigma_B^2) + \sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2\sigma_B^2 + \sigma_A^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N} - \frac{\mu^2\sigma_A^2 + \sigma_B^2\sigma_B^2}{\sigma_A^2\sigma_B^2 N}\\
&= -\frac{1}{N}.
\end{aligned}
$$

The end result $-1/N$ is of the same order of magnitude as the original terms. Therefore $\lambda_0^* = (-1/N)(1 + O(\epsilon))$. Similarly

$$
\lambda_a^* \doteq -\frac{1}{\sigma_A^2 N_{i\bullet} N} N\mu^2 + \frac{1}{\sigma_A^2 N_{i\bullet}^2} N_{i\bullet}(\mu^2 + \sigma_A^2) = -\frac{\mu^2}{\sigma_A^2 N_{i\bullet}} + \frac{\mu^2 + \sigma_A^2}{\sigma_A^2 N_{i\bullet}} = \frac{1}{N_{i\bullet}}
$$

and

$$
\lambda_b^* \doteq \frac{1}{N_{\bullet j}},
$$

and both of these approximations involve multiplication by $1 + O(\epsilon)$. In this limit then

$$
\hat{Y}_{ij} = \overline{Y}_{i\bullet}(1 + O(\epsilon)) + \overline{Y}_{\bullet j}(1 + O(\epsilon)) - \overline{Y}_{\bullet\bullet}(1 + O(\epsilon))
$$

which make intuitive sense as $(\hat{\mu} + \hat{a}_i) + (\hat{\mu} + \hat{b}_j) - \hat{\mu}$.

20 Smoothing predictors

In some cases we may want a better estimate of $\mathbb{E}(Y_{ij})$ than Y_{ij} itself is. Such a predictor could take the form

$$\hat{Y}_{ij} = \hat{Y}_{ij}(\lambda) = \lambda_0 \sum_{rs} Z_{rs}Y_{rs} + \lambda_a \sum_{s} Z_{is}Y_{is} + \lambda_b \sum_{r} Z_{rj}Y_{rj} + \lambda_{ab}Z_{ij}Y_{ij}. \tag{117}$$

It puts either extra or reduced weight on Y_{ij} itself, depending on the sign of λ_{ab}. This predictor is only useful when $Z_{ij} = 1$, so it does not apply in the new row or new column cases either. It is only nontrivial when our goal is to estimate $\mu + a_i + b_j$, not Y_{ij} itself. So we only consider $\widetilde{L} = \mathbb{E}((\hat{Y}_{ij} - \mu - a_i - b_j)^2)$ here.

Lemma 5. *The MSE for the linear predictor* (117) *is*

$$\begin{aligned}
\widetilde{L} = \mu^2\bigl(1-\lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\bigr)^2 + \sigma_A^2 + \sigma_B^2 + \lambda_0^2\Bigl(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Bigr)\\
+ \lambda_a^2\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Bigr) + \lambda_b^2\Bigl(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr)\\
- 2\lambda_0\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Bigr) - 2\lambda_a\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij}\Bigr) - 2\lambda_b\Bigl(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j}\Bigr)\\
+ 2\lambda_0\lambda_a\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Bigr) + 2\lambda_0\lambda_b\Bigl(\sigma_A^2 \sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr)\\
+ 2\lambda_a\lambda_b Z_{ij}\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Bigr).
\end{aligned}$$

Proof. This problem only arises when $Z_{ij} = 1$, which we assume for the rest of this section. Then

$$\begin{aligned}
\mathbb{E}((\tilde{Y}_{ij} - (\mu + a_i + b_j))^2) &= \mathbb{E}((\hat{Y}_{ij} + \lambda_{ab}Y_{ij} - (\mu + a_i + b_j))^2)\\
&= \widetilde{L} + \lambda_{ab}^2\mathbb{E}(Y_{ij}^2) + 2\lambda_{ab}\mathbb{E}(Y_{ij}\hat{Y}_{ij}) - 2\lambda_{ab}\mathbb{E}(Y_{ij}(\mu + a_i + b_j)).
\end{aligned}$$

Now $\mathbb{E}(Y_{ij}(\mu + a_i + b_j)) = \mu^2 + \sigma_A^2 + \sigma_B^2$ and $\mathbb{E}(Y_{ij}\hat{Y}_{ij}) = \mu^2(N\lambda_0 + N_{i\bullet}\lambda_a + N_{\bullet j}\lambda_b) + \mathrm{Cov}(Y_{ij}, \hat{Y}_{ij})$ for

$$\begin{aligned}
\mathrm{Cov}(Y_{ij}, \hat{Y}_{ij}) &= \mathrm{Cov}(Y_{ij}, \lambda_0 Y_{\bullet\bullet} + \lambda_a Y_{i\bullet} + \lambda_b Y_{\bullet j})\\
&= \lambda_0\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\bigr) + \lambda_a\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2 Z_{ij}\bigr) + \lambda_b\bigl(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2 Z_{ij}\bigr)\\
&= \lambda_0\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\bigr) + \lambda_a\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 Z_{ij} + \sigma_E^2\bigr) + \lambda_b\bigl(\sigma_A^2 Z_{ij} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\bigr),
\end{aligned}$$

since we assume that $Z_{ij} = 1$. Therefore $\mathbb{E}((\tilde{Y}_{ij} - \mu - a_i - b_j)^2)$ equals

$$\begin{aligned}
&\mu^2\bigl(1-\lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j}\bigr)^2 + \sigma_A^2 + \sigma_B^2 + \lambda_0^2\Bigl(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Bigr)\\
&+ \lambda_a^2\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Bigr) + \lambda_b^2\Bigl(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr)\\
&- 2\lambda_0\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Bigr) - 2\lambda_a\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2\Bigr) - 2\lambda_b\Bigl(\sigma_A^2 + \sigma_B^2 N_{\bullet j}\Bigr)\\
&+ 2\lambda_0\lambda_a\Bigl(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is}N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Bigr) + 2\lambda_0\lambda_b\Bigl(\sigma_A^2 \sum_r Z_{rj}N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Bigr)\\
&+ 2\lambda_a\lambda_b\Bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Bigr) + \lambda_{ab}^2(\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2)\\
&+ 2\lambda_{ab}\Bigl(\lambda_0\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\bigr) + \lambda_a\bigl(\sigma_A^2 N_{i\bullet} + \sigma_B^2 + \sigma_E^2\bigr) + \lambda_b\bigl(\sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2\bigr)\Bigr)\\
&- 2\lambda_{ab}(\mu^2 + \sigma_A^2 + \sigma_B^2) + 2\mu^2\lambda_{ab}(N\lambda_0 + N_{i\bullet}\lambda_a + N_{\bullet j}\lambda_b).
\end{aligned}$$

Gathering up the coefficient of μ^2 we get

$$\mu^2\bigl(1-\lambda_0 N - \lambda_a N_{i\bullet} - \lambda_b N_{\bullet j} - \lambda_{ab}\bigr)^2 + \sigma_A^2 + \sigma_B^2 + \lambda_0^2\Bigl(\sigma_A^2 \sum_r N_{r\bullet}^2 + \sigma_B^2 \sum_s N_{\bullet s}^2 + \sigma_E^2 N\Bigr)$$

$$
\begin{aligned}
&+ \lambda_a^2\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 N_{i\bullet} + \sigma_E^2 N_{i\bullet}\Big) + \lambda_b^2\Big(\sigma_A^2 N_{\bullet j} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
&- 2\lambda_0\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j}\Big) - 2\lambda_a\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2\Big) - 2\lambda_b\Big(\sigma_A^2 + \sigma_B^2 N_{\bullet j}\Big)\\
&+ 2\lambda_0\lambda_a\Big(\sigma_A^2 N_{i\bullet}^2 + \sigma_B^2 \sum_s Z_{is} N_{\bullet s} + \sigma_E^2 N_{i\bullet}\Big) + 2\lambda_0\lambda_b\Big(\sigma_A^2 \sum_r Z_{rj} N_{r\bullet} + \sigma_B^2 N_{\bullet j}^2 + \sigma_E^2 N_{\bullet j}\Big)\\
&+ 2\lambda_a\lambda_b\Big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\Big) + \lambda_{ab}^2(\sigma_A^2 + \sigma_B^2 + \sigma_E^2)\\
&+ 2\lambda_{ab}\Big(\lambda_0\big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big) + \lambda_a\big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 + \sigma_E^2\big) + \lambda_b\big(\sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big)\Big)\\
&- 2\lambda_{ab}(\sigma_A^2 + \sigma_B^2).
\end{aligned}
$$

Half of the derivative of this squared error with respect to λ_{ab} is

$$
\begin{aligned}
&\lambda_{ab}(\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2)\\
&\quad + \lambda_0\big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big) + \lambda_a\big(\sigma_A^2 N_{i\bullet} + \sigma_B^2 + \sigma_E^2\big) + \lambda_b\big(\sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big)\\
&\quad - \mu^2 - \sigma_A^2 - \sigma_B^2 + \mu^2(N\lambda_0 + N_{i\bullet}\lambda_a + N_{\bullet j}\lambda_b).
\end{aligned}
$$

We see that given the other λ choices, this derivative is decreasing at 0 (hence we favor positive self-weight) if

$$
\mu^2+\sigma_A^2+\sigma_B^2 > \lambda_0\big(N\mu^2+\sigma_A^2 N_{i\bullet}+\sigma_B^2 N_{\bullet j}+\sigma_E^2\big)+\lambda_a\big(N_{i\bullet}\mu^2+\sigma_A^2 N_{i\bullet}+\sigma_B^2+\sigma_E^2\big)+\lambda_b\big(N_{\bullet j}\mu^2+\sigma_A^2+\sigma_B^2 N_{\bullet j}+\sigma_E^2\big).
$$

Furthermore, the optimal self-weight, given the other λ's is

$$
\begin{aligned}
&\frac{1}{\mu^2 + \sigma_A^2 + \sigma_B^2 + \sigma_E^2} \times \Big(\mu^2 + \sigma_A^2 + \sigma_B^2 - \lambda_0\big(N\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big)\\
&\qquad - \lambda_a\big(N_{i\bullet}\mu^2 + \sigma_A^2 N_{i\bullet} + \sigma_B^2 + \sigma_E^2\big) - \lambda_b\big(N_{\bullet j}\mu^2 + \sigma_A^2 + \sigma_B^2 N_{\bullet j} + \sigma_E^2\big)\Big).
\end{aligned}
$$

□

The point of this predictor is that we might expect another observation to be made later in row i and column j. Then estimating $\mu + a_i + b_j$ is a better way to predict than repeating the earlier Y_{ij}. To use Lemma 5 after a second pass, one can compute $\widetilde{L}$ as the given quadratic function in the four variables λ_0, λ_a, λ_b and λ_{ab}. The minimizer of that quadratic gives weights to apply in prediction. When σ_E^2 is very small then Y_{ij} is already close to $\mu + a_i + b_j$ and placing special weight on Y_{ij} will be advantageous.