QMC for unbounded integrands

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Setting for QMC

\[ I = \int_{(0,1)^d} f(x) \, dx \]

\[ \hat{I} = \frac{1}{n} \sum_{i=1}^{n} f(x_i) \]

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QMC for unbounded integrands
Koksma-Hlawka inequality

\[ |\hat{I} - I| \leq D_n^*(x_1, \ldots, x_n) \times \| f \|_{HK} \]

\[ D_n^*(\cdot) = \text{star discrepancy} \]

\[ \| \cdot \|_{HK} = \text{total variation, Hardy & Krause} \]

Rates and bounds

\[ D_n^*(\cdot) = O(n^{-1} \log(n)^{d-1}) \text{ attainable} \]

Then \[ |\hat{I} - I| = O(n^{-1+\epsilon}) \]

Beating MC \quad \text{RMSE} = O(n^{-1/2})

Unless:

\[ \| f \|_{HK} = \infty \implies |\hat{I} - I| \leq \infty. \]

QMC for unbounded integrands
Infinite HK variation

Examples

- Indicator (characteristic) functions of sets \( d \geq 2 \)
- Piecewise functions, e.g. \( \max(g_1(x), g_2(x)) \) \( d \geq 3 \)
- Unbounded functions

Sources of unbounded functions

- Feynmann diagrams (Kinoshita) in physics
- Unbounded domains mapped to \((0, 1)^d\)
  e.g. \( f(\Phi^{-1}(x)) \) for Gaussian CDF \( \Phi(\cdot) \)
- Importance sampling
  e.g. \( \int f(x)q(x)dx = \int f(x)\frac{q(x)}{p(x)}p(x)dx \)
Example unbounded integrands

- $-\log(1-x)$
- $\exp(\Phi^{-1}(x))$
- $(1-x)^{3/4}$

QMC for unbounded integrands
Classical techniques

Avoid the singularity

Midpoint rules $1/n, \ldots, (n - 1)/n$

Other “open” rules Davis & Rabinowitz

Monte Carlo

Subtract the singularity

$\tilde{f}(x) = f(x) - s(x)$

$\tilde{f}(x)$ bounded and $\int s(x)\,dx$ known

Use $\int s(x)\,dx$ plus quadrature on $\tilde{f}$

Avoidance is usually simpler

QMC for unbounded integrands
van der Corput points

\[ x_0 = 0 \quad x_1 = 1/2 \quad x_2 = 1/4 \quad x_3 = 3/4 \quad x_4 = 1/8 \quad x_5 = 5/8 \quad \cdots \]

Avoid singularity at \( x = 1 \)

For \( f(x) = -\log(1 - x) \), looks like \( \hat{I}_n \to I = 1 \)
Error vs $n$

- $n^{-1/2}$
- $n^{-1}$
- $\log(n)/n$
- $\log(n)^2/n$

- marks $n = 2^m$
- RQMC RMSE

QMC for unbounded integrands
van der Corput and \(- \log(1 - x)\)

For \(n = 2^m\)

\[ x_i \text{ are } 0/n \quad 1/n \quad 2/n \cdot \cdot \cdot (n - 1)/n \]

\[ \int_0^{1-1/n} f(x)dx \leq \hat{I}_n \leq \int_0^1 f(x)dx \]

\[ 1 - 1/n - \log(n)/n \leq \hat{I}_n \leq 1 \]

so \( |\hat{I}_n - I| = O(\log(n)/n) = O(n^{-1+\epsilon}) \)

Left endpoint rule

by monotonicity

calculus

for \(n = 2^m\)
For general $n$

Consider $n = 100 = 1100100_{(2)}$

$x_1, \ldots, x_{64}$ a left endpoint rule

Next 32 points shifted left endpoint rule

Final 4 points shifted left endpoint rule

Generally: one rule for every binary 1 in $n$

At most: $\log_2(n + 1)$ shifted rules

Ultimately:

\[
1 \geq \hat{I}_n \geq 1 - \log_2(n + 1) \left( \frac{1}{n} + \frac{\log(n)}{n} \right)
\]

\[
|\hat{I}_n - I| = O\left(\frac{\log(n)^2}{n}\right) = O\left(n^{-1+\epsilon}\right)
\]
Randomized van der Corput

For \( n = 2^m \)

A scrambled \((0, m, 1)\)-net

Don’t get \( \text{RMSE} = O(n^{-3/2}) \) \( \cdots \) because \( \int f'(x)^2 dx = \infty \)

\( \max_i f(x_i) \) has variance 1 (indep of \( n \))

\( k' \)th largest \( f(x_i) \) has variance \( 1/k^2 + O(1/k^3) \) (indep of \( n \))

\[ E((\hat{I}_n - I)^2) \doteq 1.0803/n^2 \quad \text{RMSE} \doteq 1.0394/n \]

For general \( n \)

\( \text{RMSE} = O(\log(n)/n) \)

scrambling only improves by \( O(\log(n)) \)
Log normal mean

\[ \varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \]

\[ \Phi(x) = \int_{-\infty}^{x} \varphi(z)dz \]

\[ f(x) = e^{\Phi^{-1}(x)} \]

\[ I = e^{1/2} \approx 1.654 \]

Also \( O(n^{-1+\epsilon}) \) (Eventually) Dashed: \( \propto n^{-0.7122} \)

Reference lines: \( n^{-1/2} \quad n^{-1} \quad \log(n)/n \quad \bullet \) marks \( n = 2^m \)

QMC for unbounded integrands
Sobol’ (1973)

Remarkable paper Covers $d = 1$ case and $d \geq 1$

For $d = 1$ & singularity at 0

Define $c_n = \min_{1 \leq i \leq n} x_i$

$$D_n^* \times \int_{c_n}^{1} |f'(x)| \, dx \to 0 \implies |\hat{I}_n - I| \to 0$$

For van der Corput (starting at $x_1 = 1/2$): $\frac{1/2}{n+1} \leq c_n \leq \frac{2}{n+1}$

For $f(x) = x^{-\beta}$ $|\hat{I} - I| = O(n^{\beta-1} \log(n))$

For $f(x) = x^{-1} \log(x)^{-\gamma}$ $|\hat{I} - I| = O(\log(n)^{1-\gamma})$
Avoiding the origin in $\mathbb{R}^d$

\[ L \]

\[ K_{\text{orig}}^\text{min}(\epsilon) \]

\[ \min_i \min_j x_{ij} \geq \epsilon \]

\[ H \]

\[ K_{\text{orig}}^\text{prod}(\epsilon) \]

\[ \min_i \prod_j x_{ij} \geq \epsilon \]

\[ C \]

\[ K_{\text{orig}}^\text{max}(\epsilon) \]

\[ \min_i \max_j x_{ij} \geq \epsilon \]

QMC for unbounded integrands
Avoiding all corners in $(0, 1)^d$

\[ K_{\text{corn}}^{\min}(\epsilon) \quad K_{\text{corn}}^{\text{prod}}(\epsilon) \quad K_{\text{corn}}^{\max}(\epsilon) \]

\[
\min_i \min_j z_{ij} \geq \epsilon \quad \min_i \prod_j z_{ij} \geq \epsilon \quad \min_i \max_j z_{ij} \geq \epsilon
\]

Where \[ z_{ij} = \min(x_{ij}, 1 - x_{ij}) \]
Sobol’ (1973) $d \geq 1$

Uses hyperbolic origin avoidance  (typo makes it look like $L$ avoidance)

$$K_n = K_{\text{orig}}^\text{prod}(\epsilon_n) \quad \epsilon_n = \min_{1 \leq i \leq n} \prod_{j=1}^{d} x_{ij}$$

**Theorem 2**

If

$$D_n^* \times \int_{K_n} \left| \frac{\partial^d f(x)}{\partial x} \right| dx \to 0,$$

and

$$\int_{(0,1)^d} x_1 x_2 \cdots x_d \left| \frac{\partial^d f(x)}{\partial x} \right| dx < \infty,$$

and similarly for $2^d - 1$ coordinate projections,

then

$$|\hat{I} - I| \to 0$$
Sobol’ (1973) continued

If $x_1, \ldots, x_{2^m}$ a $(\tau, m, d)$–net (from LP$_\tau$-sequence)

Then $x_i \in K_{\text{prod}}^{\text{orig}}(2^{d-\tau}/n)$

For $f(x_0) = x_0^{-\beta_1} x_0^{-\beta_2} \ldots x_0^{-\beta_d}$ $x_0 \in (0, 1)^d$

$|\hat{I} - I| \to 0$

if all $\beta_j < 1$
After Sobol’ (1973)

Hickernell & Sloan & Wasilkowski

Thorough study of tractability over unbounded domains
Singlarities removed by transformation
See also Mathé & Wei and Genz & Monahan

Klinger article and dissertation

Among \( x_1, \ldots, x_n \) two Halton points cannot be in the same small box
Same for digital nets with \( t = 0 \)

After deleting point at origin, remainder are in \( K^\text{max}_{\text{orig}} \)

Kronecker points \( x_{ij} = \sqrt{p_j} i \mod 1 \) are in \( K^\text{prod}_{\text{orig}}(\gamma n^{1-\delta}) \)

Hartinger & Kainhofer & Tichy

Generalize to \( \int h(x)f(x)dx \) with \( f \) singular at origin and \( h \) non-uniform

Sample with low \( h \) discrepancy, adapting Hlawka & Möck

De Doncker & Guan

Accelerate convergence on singularities

QMC for unbounded integrands
Three epsilon approach

First extend $f \cdots$ details to follow

$$f_\epsilon(x) = \tilde{f}(x) = \begin{cases} f(x), & x \in K(\epsilon) \\ ???, & x \notin K(\epsilon) \end{cases}$$

Then

$$|\hat{I} - I| \leq |\hat{I} - \hat{I}_\epsilon| + |\hat{I}_\epsilon - I_\epsilon| + |I_\epsilon - I|$$

$$I_\epsilon = \int f_\epsilon(x) dx$$

$$\hat{I}_\epsilon = \frac{1}{n} \sum_{i=1}^{n} f_\epsilon(x_i)$$

QMC for unbounded integrands
Analysis

Suppose \( x_1, \ldots, x_n \in K(\epsilon_n) \) with \( \epsilon = \epsilon_n \to 0 \)

\[
\hat{I} - \hat{I}_\epsilon = \frac{1}{n} \sum_{i=1}^{n} f(x_i) - f_\epsilon(x_i) = 0
\]

\[
|I - I_\epsilon| \leq \int_{K^c} |f(x) - f_\epsilon(x)| \, dx
\]

\[
|\hat{I}_\epsilon - I_\epsilon| \leq D_n^* \times \|f_\epsilon\|_{HK}
\]

So that:

\[
|\hat{I} - I| \leq \int_{K^c} |f(x) - f_\epsilon(x)| \, dx + D_n^* \times \|f_\epsilon\|_{HK}
\]

\[
= \text{Truncation error} + \text{Quadrature error}
\]

We need \( f_\epsilon \approx f \) with small variation

QMC for unbounded integrands
Extendable regions

Region $K$ with anchor $c \in K$

And $x \in K \implies \text{Rect}[x, c] \subseteq K$
Sobol’ extension

Assume $\partial^d f / \partial x$ exists (extendable function)

For $d = 1$

\[ f(z) = f(c) + \int_c^z f'(x) \, dx \quad \text{with} \quad \int_c^z = - \int_z^c \]

Then

\[ \tilde{f}(z) = f(c) + \int_c^z f'(x) 1_{x \in K} \, dx \]

For $d = 2$

\[ \tilde{f}(z) = f(c) + \int_{c_1}^{z_1} \frac{\partial f(x_1, c_2)}{\partial x_1} 1_{(x_1, c_2) \in K} \, dx_1 + \int_{c_2}^{z_2} \frac{\partial f(c_1, x_2)}{\partial x_2} 1_{(c_1, x_2)} \]

\[ + \int_{c_1}^{z_1} \int_{c_2}^{z_2} \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} 1_{x \in K} \, dx_2 \, dx_1 \]

For $d \geq 1$

\[ \tilde{f}(z) = \sum_{u \subseteq \{1, \ldots, d\}} \int_{[c_u, z_u]} \frac{\partial^{|u|} f(x_u; c_u)}{\partial x_u} 1_{x_u; c_u \in K} \, dx_u \]
Vitali variation of $\tilde{f}$

\[
\frac{\partial^d f(x)}{\partial x} = 0, \quad x \notin K, \quad \text{and,}
\]

\[
\|\tilde{f}\|_{\text{Vitali}} \leq \int_K \left| \frac{\partial^d f(x)}{\partial x} \right| dx
\]

Could hardly be lower!

Sobol’ (1973) must have known \quad \text{(proof not published)}

Hardy & Krause variation of $\tilde{f}$

If $c = (1, \ldots, 1)$ use $2^d - 1$ “slices” $f(x_u : c-u)$

Else chop $(0, 1)^d$ into to $2^d$ subcubes
Halton sequences

Radical inversion

\[ i = \cdots a_3 a_2 a_2 a_1 \quad \text{base } b \]

\[ \phi_b(i) = 0.a_1 a_2 a_3 \cdots \quad \text{base } b \]

van der Corput

\[ x_i = \phi_2(i) \]

Halton

\[ x_i = (\phi_{p_1}(i), \phi_{p_2}(i), \ldots, \phi_{p_d}(i)) \]

\[ p_j \quad \text{relatively prime, usually first } d \text{ primes} \]
First few Halton points

Appear to avoid origin, if we start at $x_1$ not $x_0 = (0, 0)$

But there are other large gaps too

QMC for unbounded integrands
Theorem

Let \( x_n = (\phi_{p_1}(n), \ldots, \phi_{p_d}(n)) \)

Where \( p_j \) are distinct primes

Then:

\[
\prod_{j=1}^{d} x_{n,j} \geq \frac{A}{n}
\]

\[
\prod_{j=1}^{d} (1 - x_{n,j}) \geq \frac{A}{n+1}
\]

\[
\prod_{j=1}^{d} \min(x_{n,j}, (1 - x_{n,j})) \geq \frac{A}{n(n+1)}
\]

where \( A = \prod_{j=1}^{d} p_j^{-1} \)

(*) Thanks to Reinhhold Kainhofer for improving the constant
Idea of proof

\[ \phi_b(i) \approx 0 \iff \phi_b(i) \text{ has many leading } 0's \quad \text{(base } b) \]

\[ \iff i \text{ has many trailing } 0's \]

But \( i \) cannot be a power of two primes \qquad \text{(if } i > 1 \text{)}

Similarly

\[ \phi_b(i) \approx 1 \iff \phi_b(i) \text{ has many leading } b - 1's \quad \text{(base } b) \]

\[ \iff i \text{ has many trailing } b - 1's \]

\[ \iff i + 1 \text{ has many trailing } 0's \]

\( i + 1 \) cannot be a power of two primes either
Proof

List primes $P_1 < P_2 < P_3 < \cdots$ put $p_j = P_{r(j)}$

First part

$$n = \prod_r P_r^{a_r} \quad a_r \text{ trailing zeros in base } P_r$$

$$\phi_{P_r}(n) \geq P_r^{-a_r-1}$$

$$\prod_{j=1}^d x_{n,j} \geq \prod_{j=1}^d p_j^{-a_{r(j)}-1} \geq \prod_{j=1}^d p_j^{-1} \prod_r P_r^{-a_r} = \frac{A}{n}$$

For the rest

$$n + 1 = \prod_r P_r^{b_r} \quad \text{and} \quad 1 - \phi_{P_r}(n) \geq P_r^{-b_r-1} \quad \ldots$$
Hyperbolic origin avoidance

\[
\min_{1 \leq i \leq n} \prod_{j=1}^{d} x_{ij} \quad \text{vs} \quad n
\]

\[d = 2, 3, 4, 5, 8, 11, 14, 17, 20\]

\[1 \leq n \leq 10^{10}\]

Reference lines are lower bounds

QMC for unbounded integrands
Hyperbolic corner avoidance

\[ \min_{1 \leq i \leq n} \prod_{j=1}^{d} \min(x_{ij}, 1 - x_{ij}) \quad \text{vs} \quad n \]

\[ d = 2, 3, 4, 5, 8, 11, 14, 17, 20 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{QMC for unbounded integrands} \]
Halton corner avoidance

\[ D_{\text{corn}}^{n,d} = \min_{1 \leq i \leq n} \prod_{j=1}^{d} \min(x_{ij}, 1 - x_{ij}) \]

\[ D_{\text{corn}}^{n,d} = O(n^{-r_d}) \quad 1 \leq r_d \leq 2 \]

Apparent rate of \( D_{\text{orig}}^{n,d} \)

<table>
<thead>
<tr>
<th>( d \in 2 : 5 )</th>
<th>1.10</th>
<th>1.24</th>
<th>1.34</th>
<th>1.39</th>
</tr>
</thead>
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<tr>
<td>( d \in 6 : 10 )</td>
<td>1.60</td>
<td>1.57</td>
<td>1.57</td>
<td>1.59</td>
</tr>
<tr>
<td>( d \in 11 : 15 )</td>
<td>1.58</td>
<td>1.51</td>
<td>1.69</td>
<td>1.66</td>
</tr>
<tr>
<td>( d \in 16 : 20 )</td>
<td>1.47</td>
<td>1.62</td>
<td>1.52</td>
<td>1.79</td>
</tr>
</tbody>
</table>

QMC for unbounded integrands
Growth conditions

Singularity at origin

\[ |f(x)| \leq B \prod_{j=1}^{d} x_j^{-A_j} \quad \text{and,} \]

\[ |\partial^u f(x)| \leq B \left( \prod_{j \in u} x_j^{-A_j} - 1 \right) \left( \prod_{j \notin u} x_j^{-A_j} \right) \]

Singularity at corners

\[ |f(x)| \leq B \prod_{j=1}^{d} \min(x_j, 1-x_j)^{-A_j} \quad \text{and,} \]

\[ |\partial^u f(x)| \leq B \left( \prod_{j \in u} \min(x_j, 1-x_j)^{-A_j - 1} \right) \left( \prod_{j \notin u} \min(x_j, 1-x_j)^{-A_j} \right) \]

For \( 0 < A_j < 1 \) and \( 0 < B < \infty \)
Avoiding L shaped regions

Theorem

If \( x_1, \ldots, x_n \in K_{\text{orig}}^{\min}(Cn^{-r}) \) respectively \( K_{\text{corn}}^{\min}(Cn^{-r}) \)
and \( f \) obeys origin (corner) growth conditions,
then

\[
|\hat{I} - I| \leq C_1 D_n^* n^r \sum_{j=1}^{d} A_j + C_2 n^r (\max_j A_j - 1)
\]

Corollary: when \( D_n^* = O(n^{-1+\epsilon}) \),

\[
|\hat{I} - I| = O(n^{-1+r} \sum_j A_j)
\]
How to avoid L shaped regions

Halton points do (after $x_0$) with $r = 1$

Linear adjustment · · ·

$$\tilde{x}_i = \epsilon_n + (1 - \epsilon_n) x_i$$ to avoid origin

$$\tilde{x}_i = \epsilon_n + (1 - 2\epsilon_n) x_i$$ to avoid corners

· · · preserves low discrepancy

$$D^*_n(\tilde{x}_1, \ldots, \tilde{x}_n) = D^*_n(x_1, \ldots, x_n) + O((1 + 2\epsilon_n)^d - 1)$$

Hlawka & Mück

$$\epsilon_n = Cn^{-1}$$ ok

QMC for unbounded integrands
Comparison to Monte Carlo

Suppose \( f(x) = \prod_{j=1}^{d} x_j^{-A_j} \quad \max_j A_j < 1 \)

QMC with \( n^{-r} \) origin avoidance has

\[
|\hat{I} - I| = O(n^{-1+r} \sum_j A_j)
\]

**MC accuracy**

\[
\max_j A_j < 1/2 \implies \int f(x)^2 dx < \infty \implies \text{MC has RMSE } O(n^{-1/2})
\]

QMC wins if \( \sum_{j=1}^{d} A_j < (2r)^{-1} \) can lose otherwise

**Extreme example**

\( \sum_j A_j > 1 \)

Low discrepancy \( x_i \) and \( \tilde{x}_i = \frac{1}{n} + (1 - \frac{2}{n})x_i \)

\( x_1 = (0, 0, \ldots, 0) \implies \tilde{x}_1 = (1/n, \ldots, 1/n) \)

\[
\hat{I} \geq n \sum A_j^{-1} \rightarrow \infty
\]
Avoiding hyperbolic regions

**Theorem**

If \( x_1, \ldots, x_n \in K_{\text{prod}}^{\text{orig}}(Cn^{-r}) \) resp. \( K_{\text{prod}}^{\text{corn}}(Cn^{-r}) \)

and \( f \) obeys origin (corner) growth conditions,

then \( |\hat{I} - I| \leq C_1 D_n^{*} n^{\eta+r} \max_j A_j + C_2 n^r (\max_j A_j - 1) \)

Any \( \eta > 0 \) and \( \eta = 0 \) if \( \max A_j \) unique

**Corollary:** for \( D_n^{*} = O(n^{-1+\epsilon/2}) \),

\( |\hat{I} - I| = O(n^{-1+\epsilon+r} \max_j A_j) \)
How to avoid hyperbolic regions

Halton points do (after $x_0$) with $1 \leq r \leq 2$

**Linear adjustment fails**

Eg $x_1 = (0, 0, \ldots, 0)$ has to move $O(n^{-r/d})$ to enter $K_{\text{prod}}^{\text{orig}}(n^{-r})$

An $n^{-1/d}$ move affects discrepancy

**Proving hyperbolic avoidance**

Requires details of $x_i$ construction

axis oriented discrepancy does not suffice

not even isotropic discrepancy ($n^{-1/d}$ again)

QMC for unbounded integrands
Comparison to Monte Carlo

If
\[ x_i \text{ in } K_{\text{corn}}^{\text{prod}}(n^{-r}) \text{ with } D_n^* = O(n^{-1+\epsilon}) \]
and
\[ f \text{ obeys growth conditions with } \max_j A_j < 1/2 \]
then QMC has
\[ |\hat{I} - I| = O(n^{-1+\epsilon+r} \max_j A_j) \]
and beats MC when
\[ \max_j A_j < (2r)^{-1} \]

Recall
\[ \sum_j A_j < (2r)^{-1} \quad \text{(for L avoidance)} \]

H avoidance closer to contours of \( f \)
Randomized quasi-Monte Carlo

Each \( x_i \sim U(0,1)^d \), \( D_n^* \leq Cn^{-1+\eta} \), \( \forall \eta > 0 \)

let \( f_\epsilon(x) = \begin{cases} f(x), & x \in K(\epsilon_n) \\ \tilde{f}(x), & x \notin K(\epsilon_n) \end{cases} \)

as before

\[
|\hat{I} - I| \leq |\hat{I} - \hat{I}_\epsilon| + |\hat{I}_\epsilon - I_\epsilon| + |I_\epsilon - I|
\]

but \( \Pr(x_i \notin K) = \text{vol}(K^c) > 0 \)

\[
E(|\hat{I} - \hat{I}_\epsilon|) \leq \frac{1}{n} \sum_{i=1}^{n} |E(f(x_i) - f_\epsilon(x_i))| = |I - I_\epsilon|
\]

\[
E(|\hat{I} - I|) \leq |\hat{I}_\epsilon - I_\epsilon| + 2|I_\epsilon - I|
\]

\[
\leq D_n^* \times \|f_\epsilon\|_{HK} + 2|I_\epsilon - I|
\]

QMC for unbounded integrands
RQMC

Doubles truncation error

But allows more $K$ choices, e.g. level sets of $f$ (if extendable)

$$E(|\hat{I} - I|) \leq \inf_K \left( D_n^* \times \|f_\epsilon\|_{HK} + 2|I_\epsilon - I| \right)$$

$$|\hat{I} - I| \leq \inf_K \left( D_n^* \times \|f_\epsilon\|_{HK} + |I_\epsilon - I| \right)$$

For RQMC $K(\epsilon_n)$ must be extendable

For QMC $K(\epsilon_n)$ must be extendable and avoided by $x_i$

Consequences

$$E(|\hat{I} - I|) = O(n^{-1+\epsilon+\max_j A_j})$$

Then RQMC beats $O(n^{-1/2})$ when $\max_j A_j < 1/2$

Can have $x_1 = (0, \ldots, 0)$ (prerandomization) e.g. lattices

QMC for unbounded integrands
Asian call and put option integrands

Obey growth conditions for any $A_j > 0$ good for (R)QMC

Like power singularity with $A_j = 0+$

but $\text{cusp} \implies \|f\|_{HK} = \infty$ even for bounded put

QMC for unbounded integrands
Summary

For unbounded $f$

1. QMC can have $|\hat{I} - I| = O(n^{-1+\epsilon}) \quad \forall \epsilon > 0$

2. or $\hat{I} \to \infty$

3. requires discrepancy plus avoidance
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