Abstract

Proebsting’s Paradox is two-stage bet where the naive Kelly gambler (wealth growth rate maximizing) can be manipulated in some disconcertingly non-optimal ways. We extend this result to $k$-stage and the limiting $\infty$-stage bets, and give a few comments.

Suppose someone offers us a bet. At some time in the future a $p$ coin is flipped, and if the coin is heads, we get $\sigma$-for-1, and 0 otherwise. How much should we invest in the bet?

In other words, let $X$ be a Bernoulli-$p$ random variable, the random wealth observed after betting $b$ proportion of total wealth is

$$S = \bar{b} + b\sigma X$$

with $\bar{b} = 1 - b$. There is sizable evidence to support choosing $b$ to maximize $E[\ln(S)]$ as the right answer, with the solution given by

$$b = \frac{p\sigma - 1}{\sigma - 1}$$

This strategy maximizes the growth rate of wealth, and protects against bankruptcy no matter what $\sigma$ and $p$ are. It goes by the name Kelly gambling. So why would you ever doubt Kelly gambling?

The short answer is you wouldn’t, most of the time.
One apparent issue has been floating around under the name Proebsting’s paradox, first encountered in an email exchange between Ed Thorp and Todd Proebsting in 2008. The idea is before the outcome of the coin flip, the same bookie offers the opportunity to wager more money, this time at odds $\sigma^2$-for-1. Investing $b_2$ more of total wealth trades $S$ for a new random variable $S_2 = \tilde{b}_2 + (b_\sigma + b_2\sigma_2)X$

with $\tilde{b}_2 = 1 - b - b_2$. Maximizing $E[\ln(S_2)]$, treating the initial investment of $b$ as fixed, gives, after a little rearranging

$$b_2 = \frac{p(\sigma_2 - \sigma)}{\sigma_2 - 1}(1 - b).$$

So as long as the bookie gives better odds than before ($\sigma_2 > \sigma$), we will bet a positive amount.

Note that we still have all the benefits of Kelly working for us. The combined bet $(b, b_2)$ gives the largest growth rate over many repetitions of the same $(\sigma, \sigma_2)$-for-1, 2 stage bet. Also we’ll never bet more than $p$ fraction of $\tilde{b}$ at the second round, and a bound for the total amount bet is

$$B_2 = b + b_2 < p + p(1 - p) \leq 1$$

with equality iff $p = 1$. We are still protected from betting all of our money, though the bound is suspiciously greater than $B_1 = b < p$.

A disconcerning consequence of the sequential bet is that the Kelly bettor will bet more if offered, say, 6-for-1 after 3-for-1 odds on a $p = .5$ bet, than if only offered the 6-for-1 odds (.475 vs .40 of wealth). In fact, this turns out to hold generally for any $p$ and $\sigma_2 > \sigma_1$. Here I’m adding the subscript to $\sigma_1$ to avoid confusion.

**Lemma 0.1.** Let $B_k(\sigma)$ denote the total Kelly bet for a $k$-stage bet with odds $\sigma = (\sigma_1, \ldots, \sigma_k)$, then for any positive expected value bet ($\sigma_2 > \sigma_1 > 1/p$)

$$B_2(\sigma_1, \sigma_2) - B_1(\sigma_2) = \frac{p(1-p)(\sigma_2 - \sigma_1)(\sigma_1 - 1/p)}{(\sigma_1 - 1)(\sigma_2 - 1)} > 0$$

(1)

**Proof.** Substituting for $(1 - b)$ gives

$$B_2 - B_1 = \frac{p(\sigma_1 - 1/p)}{\sigma_1 - 1} + \frac{p(\sigma_2 - \sigma_1)(1 - p)\sigma_1}{(\sigma_2 - 1)(\sigma_1 - 1)} - \frac{p(\sigma_2 - 1/p)}{\sigma_2 - 1}$$

After combining fractions the numerator simplifies to that of (1). \qed
Why does this happen? One explanation [wikipedia] is that the bettor’s share of the $\sigma_1$-for-1 bet devalues in the presence of the $\sigma_2$-for-1 bet, much in the same way an investor feels less wealthy when the price of an owned stock goes down. The bettor still bets the optimal Kelly amount $B_1(\sigma_2)$ at the second stage, but now with respect to an implied total wealth less than 1.

Specifically, the $b\sigma_1 X$ variable can be replicated at the second stage by setting $b_2 = b(\sigma_1/\sigma_2)$. Since the $b$’s can be considered the deterministic cost of the random variable, and $\sigma_2 > \sigma_1$, the value of the original bet drops from $b$ to $b_2$, and the betters total wealth in the second stage falls to $(1 - b) + b_2 = 1 - b(1 - \sigma_1/\sigma_2)$.

Even though the explanation is logical, the result still makes us feel uneasy. Somehow the nearsighted Kelly gambler isn’t able to protect his wealth from a (possibly) manipulating bookie.

These results extend, and become even more extreme, in the general $k$-stage bet, and especially so for the limiting $\infty$-stage bet. We start with an equation for $B_k(\sigma)$.

**Lemma 0.2.** For a general $N$-stage bet with $\sigma_i > 1$ for all $i$, Kelly betting at each stage invests

$$b_k = p\alpha_k \bar{b}_k$$

proportion of total wealth with cumulative proportion invested

$$B_k = \sum_{i=1}^{k} p\alpha_i (1 - p\alpha_{i+1}) \ldots (1 - p\alpha_k)$$

where $\alpha_k = \frac{\sigma_k - \sigma_{k-1}}{\sigma_{k-1}}$ and $\bar{b}_k = 1 - \sum_{i=1}^{k-1} b_i$, the proportion of remaining wealth.

**Proof.** The first order condition for $\mathbb{E}[\ln(S_2)]$ holding $b$ fixed gives the optimal $b_2$.

$$b^*_2 = \frac{p\sigma_2 (1 - b) - q\sigma_1 b - (1 - b)}{\sigma_1 - 1}$$

Adding and subtracting $p\sigma_1 (1 - b)$ in the numerator gives $b_2$ as in (2) plus a remainder that can be shown to equal 0.

At any subsequent stage $k$,
\[ S_k = \bar{b}_k + \left( \sum_{i=1}^{k} b_i \sigma_i \right) X. \]

Since all previous bets are fixed, \( S_k \) is of the same form as \( S_2 \), but with updated constants, \( \bar{b} \rightarrow \bar{b}_k \) and \( b\sigma \rightarrow \sum_{i=1}^{k-1} b_i \sigma_i \), and hence the first order condition as well. The form \((2)\), is found similarly to the case \( k = 2 \). For the first stage, \( b_1 \equiv b \), can also be written in the form of \((2)\) with \( \sigma_0 = 1/p \) and \( \bar{b}_0 = 1 \).

Equation \((3)\) now follows as \( B_k \) satisfies the following recursive relation \( B_k = (1 - p\alpha_k)B_{k-1} + p\alpha_k \) with initial term \( B_1 = p\alpha_1 \).

Lemma 0.2 gives one interpretation of a Kelly bettor’s strategy. At each stage, they invest more of their wealth if better odds are offered over the previous stage, and the amount of remaining wealth invested is proportional to \( p \) times a factor penalizing odds increases over already large odds. Note that \( 0 \leq \alpha_k \leq 1 \), with equality only in the cases of no change in odds and infinite odds. The total proportion invested is harder to describe, but does set us up to talk about Proebsting’s paradox for many stages.

First a corrolary on bounds,

**Corollary 0.3.** At each stage the Kelly bettor will not bet more than \( p \) proportion of remaining wealth, and using this bound

\[ B_k < c_k \to 1. \]

Furthermore proportional Kelly gambling does not decrease the bound.

**Proof.** Extending the result for 2 stages it is straightforward to show \( b_i < p(1 - p)^{i-1} \) and hence

\[ c_k = \sum_{i=1}^{k} p(1 - p)^{i-1} \to p \sum_{i=0}^{\infty} (1 - p)^i = 1 \]

For proportional Kelly the bound is now \( b_i < \gamma p(1 - \gamma p)^{i-1} \) with \( \gamma \in (0,1) \) and the same infinite sum gives \( c'_k \to 1 \).

Finally, I’ll give 2 examples showing both a case of \( B_\infty = 1 \) and \( B_\infty < 1 \).
Example 0.4. Suppose the odds are increasing at a constant rate, \( \alpha_k = \alpha \) for all \( k \) and \( \alpha \in (0, 1) \). Then

\[
B_k = p\alpha \sum_{i=1}^{k} (1 - p\alpha)^{k-i} \rightarrow 1
\]

This is essentially the case of the original Proebsting paradox, \( \sigma_k = 2^k \), where \( \alpha \approx .5 \) for large \( k \).

Example 0.5. Suppose the odds are increasing, but at a decreasing rate, \( \alpha_k = 2^{-k} \), then a really quick bound is

\[
B_k = \sum_{i=1}^{k} p2^{-i} \prod_{j=i+1}^{k} (1 - p2^{-j}) < \sum_{i=1}^{k} p2^{-i} \rightarrow p \quad (4)
\]

and taking 1 more term in the expansion gives

\[
B_k < \sum_{i=1}^{k} p2^{-i}(1 - p2^{-i-1}) = p - \frac{p^2}{6}.
\]

Note that for either expansion to hold, we need \( p < 1 \) because if \( p = 1 \) the Kelly bettor takes \( b_1 = 1 \) and hence all other \( b_i = 0 \), though the bound (4) in this case still holds.

It seems as though a Kelly bettor can find themselves risking most of their wealth, even on low probability bets, as long as the odds are increasing fast enough. One might even imagine an example of Proebsting’s paradox in action where a real estate developer invests more and more into a property as housing prices bubble, even though the underlying (known!) sale probability may be low.

One solution is of course never to bet more than \( \rho \) proportion of your wealth. It is also not hard to imagine a slightly more savvy bettor incorporating prior knowledge on the chance of future stages, and making assumptions on the stochastic nature of odds to get a random bet process, \( b_k(\sigma) \).