Last Time

- Definition and properties of expectation
- Integrability
- Inequalities (Jensen, Markov, Schwarz)

Today’s lecture: Sections 1.3.1, 1.4.1
Law of a RV

- Let $X$ be a RV on $(\Omega, \mathcal{F}, \mathbb{P})$. The **law (aka distribution)** of $X$ is the probability measure $\mathbb{P}_X$ on $(\mathbb{R}, \mathcal{B})$ defined for $B \in \mathcal{B}$ as

  $$\mathbb{P}_X(B) \doteq \mathbb{P}(X \in B)$$

- $\mathbb{P}_X$ determines the values $\mathbb{P}$ on $\sigma(X)$
Distribution Function of a RV

- Let $X$ be a RV on $(\Omega, \mathcal{F}, \mathbb{P})$. The (cumulative) distribution function of $X$ is the function $F_X : \mathbb{R} \to [0, 1]$ defined for $x \in \mathbb{R}$ as

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}_X((-\infty, x])$$

- The distribution function $F_X$ uniquely determines the law $\mathbb{P}_X$ of $X$

- Any RV has a distribution function

- Special cases:
  - Discrete RV: $F_X(x) = \sum_{i: x_i \leq x} \mathbb{P}(X = x_i)$
  - Abs cont RV: $F_X(x) = \int_{-\infty}^{x} f_X(u) du$. $F_X$ is continuous and $\frac{dF_X}{dx}(x)$ exists and equals $f_X(x)$ for almost every $x$
Properties of Distribution Function

• Let $X$ be a RV with distribution function $F_X$. Then
  ◦ $F_X$ is nondecreasing
  ◦ $F_X$ is right-continuous, i.e. $\lim_{h \downarrow 0} F_X(x + h) = F_X(x)$
  ◦ $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

• *(Inversion)* Let $F$ be a function satisfying the three properties above. Then there exists some probability space $(\Omega, \mathcal{F}, P)$ and a real-valued RV $X$ defined on it such that $F$ is the distribution function of $X$
  ◦ Key to proof: consider $[0, 1]$ with the Borel $\sigma$-field and the uniform probability measure
  ◦ Define the “inverse" of $F$

$$\varphi(u) \doteq \inf\{x : F(x) \geq u\}$$
Change of Variables

• Let $X$ be a RV on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function.

• If $g$ is nonnegative or $\mathbb{E}|g(X)| < \infty$ then

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}} g(x) d\mathbb{P}_X(x)$$

• Special cases:
  ◦ Discrete RV: $\mathbb{E}(g(X)) = \sum_i g(x_i) \mathbb{P}(X = x_i)$
  ◦ Absolutely continuous RV: $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$
Almost Sure Convergence

- Let $X_1, X_2, \ldots$ be a sequence of RV’s on $(\Omega, \mathcal{F}, IP)$ and let $X$ be another RV on this space.

- $X_n$ converges to $X$ almost surely as $n \to \infty$ if there exists $A \in \mathcal{F}$ with $IP(A) = 1$ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in A$.

- If $X_n \to X$ a.s. and $f$ is continuous then $f(X_n) \to f(X)$ a.s.

- Strong Law of Large Numbers: if $\{X_n\}$ is an i.i.d. sequence with finite mean $\mu$ then

$$\frac{\sum_{i=1}^{n} X_i}{n} \to \mu \text{ a.s. as } n \to \infty$$
Complete Probability Space

- We say that \((\Omega, \mathcal{F}, \mathbb{P})\) is a **complete probability space** if \(B \in \mathcal{F}\) with \(\mathbb{P}(B) = 0\) implies that \(N \in \mathcal{F}\) for any \(N \subset B\).

- Any probability space can be completed by adding to \(\mathcal{F}\) all subsets of sets of probability 0.

- *We will always assume the probability space is complete*

- Completeness guarantees that an a.s. limit of a RV is itself an RV.
Convergence in Probability

• Let $X_1, X_2, \ldots$ be a sequence of RV’s on $(\Omega, \mathcal{F}, IP)$ and let $X$ be another RV on this space.

• $X_n$ converges to $X$ in probability as $n \to \infty$ if for any $\epsilon > 0$,

\[
\lim_{n \to \infty} IP(|X_n - X| > \epsilon) = 0
\]
Convergence in Probability & A.S. Convergence

- If $X_n \rightarrow X$ a.s. then $X_n \rightarrow X$ in probability
- If $X_n \rightarrow X$ in probability then there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \rightarrow X$ a.s. as $k \rightarrow \infty$
Borel-Cantelli Lemmas

• Let \( A_k \in \mathcal{F}, k = 1, 2, \ldots \) and define

\[
A^\infty = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k
\]

• \( A^\infty \) is the set of outcomes that occur infinitely often (i.o.)

• First BC Lemma:
  If \( \sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty \) then \( \mathbb{P}(A^\infty) = 0 \)

• Second BC lemma:
  If \( \{A_k\} \) are independent and \( \sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty \) then
  \( \mathbb{P}(A^\infty) = 1 \)
Example using BC Lemmas

- Let $X_1, X_2, \ldots$ be i.i.d. exponential RV’s with rate 1, i.e.
  $\mathbb{P}(X > x) = e^{-x}$

- Show that
  $$\mathbb{P}(X_n > \alpha \log n \text{ for infinitely many } n) = \begin{cases} 
  0, & \alpha > 1, \\
  1, & \alpha \leq 1 
  \end{cases}$$

- Define
  $$L = \limsup_{n \to \infty} \frac{X_n}{\log n}$$

  and show that $L = 1$ a.s.