Last Time

- Introduction
- Measurable space
- Generated $\sigma$-fields
- Borel $\sigma$-field

Today’s lecture: Sections 1.1–1.2.2
Probability space

- A **probability space** is a triple \((\Omega, \mathcal{F}, \mathbb{P})\), where \((\Omega, \mathcal{F})\) is a measurable space and \(\mathbb{P}\) is a probability measure.

- A **probability measure** is a set function \(\mathbb{P} : \mathcal{F} \rightarrow [0, 1]\) which satisfies:
  - \(\mathbb{P}(\Omega) = 1\)
  - \(0 \leq \mathbb{P}(A) \leq 1\) for all \(A \in \mathcal{F}\)
  - **Countable additivity**: if \(A_i \in \mathcal{F}, i = 1, 2, \ldots\) are mutually disjoint (i.e. \(A_i \cap A_j = \emptyset, i \neq j\)) then
    \[
    \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)
    \]

- If \(\mathbb{P}(A) = 1\) then we say \(A\) occurs **almost surely** (a.s.)
Specifying the Probability Measure \( \mathcal{IP} \)

- **Countable \( \Omega \)**
  - Set \( \mathcal{F} = 2^\Omega \)
  - Define \( p_\omega \) for each \( \omega \in \Omega \), such that \( 0 \leq p_\omega \leq 1 \) and \( \sum_{\omega \in \Omega} p_\omega = 1 \)
  - Then \( \mathcal{IP}(A) = \sum_{\omega \in A} p_\omega \) defines a probability measure on \((\Omega, 2^\Omega)\)

- **Uncountable \( \Omega \)**
  - Consider a set of generators \( \{A_\alpha : \alpha \in \Gamma\} \) with \( \mathcal{F} = \sigma(\{A_\alpha : \alpha \in \Gamma\}) \)
  - Define a probability measure \( \mathcal{IP}(A_\alpha) \) for all \( A_\alpha, \alpha \in \Gamma \)
  - Then (under mild conditions) \( \mathcal{IP} \) extends uniquely to a probability measure on \((\Omega, \mathcal{F})\) (see Rosenthal, Section 2.3)
Some properties of $\mathbb{P}$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B, A_i, B_i \in \mathcal{F}$, 
$i = 1, 2, \ldots$

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- **Monotonicity**: if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$
- **Countable subadditivity**: if $A \subseteq \bigcup_{i=1}^{\infty} A_i$ then 
  $\mathbb{P}(A) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$
- **Continuity from below**: if $A_i \uparrow A$ ($A_1 \subset A_2 \subset \cdots$ and 
  $\bigcup_{i=1}^{\infty} A_i = A$) then $\mathbb{P}(A_i) \uparrow \mathbb{P}(A)$
- **Continuity from above**: if $B_i \downarrow B$ ($B_1 \supset B_2 \supset \cdots$ and 
  $\bigcap_{i=1}^{\infty} B_i = B$) then $\mathbb{P}(B_i) \downarrow \mathbb{P}(B)$
**Definition of Random Variable**

- A **random variable** is a real-valued $\mathcal{F}$-measurable function on $(\Omega, \mathcal{F})$.
- That is, $X : \Omega \to \mathbb{R}$ satisfies
  \[
  X^{-1}(B) = \{ \omega : X(\omega) \in B \} \in \mathcal{F}, \text{ for all } B \in \mathcal{B}.
  \]
  Equivalently,
  \[
  X^{-1}((-\infty, \alpha]) = \{ \omega : X(\omega) \leq \alpha \} \in \mathcal{F}, \text{ for all } \alpha \in \mathbb{R}.
  \]
- Special case: $(\Omega, \mathcal{F}) = (\mathbb{R}^n, \mathcal{B}^n)$. A real-valued $\mathcal{B}^n$-measurable function on $(\mathbb{R}^n, \mathcal{B}^n)$ is called **Borel-measurable** (or a Borel function).
- Notation: often write $\{ \omega : X(\omega) \in B \}$ as $\{ X \in B \}$. 

MATH136/STAT219 Lecture 2, September 24, 2008 – p. 5/14
Definition of Random Vector

- A random vector is an $\mathbb{IR}^n$-valued $\mathcal{F}$-measurable function on $(\Omega, \mathcal{F})$

- That is, $X = (X_1, \ldots, X_n) : \Omega \to \mathbb{IR}^n$ satisfies

  $$X^{-1}(B) \doteq \{\omega : X(\omega) \in B\} \in \mathcal{F}, \text{ for all } B \in \mathcal{B}^n$$

  Equivalently, for all $\alpha_i \in \mathbb{IR}, i = 1, \ldots, n$,

  $$\{\omega : X_1(\omega) \leq \alpha_1, \ldots, X_n(\omega) \leq \alpha_n\} \in \mathcal{F}$$

- Note: $X = (X_1, \ldots, X_n)$ is a random vector if and only if $X_i$ is a random variable for each $i = 1, \ldots, n$
Simple Functions

- **Indicator function (RV) of a set:**

\[
I_A(\omega) = \begin{cases} 
1, & \omega \in A \\
0, & \omega \notin A 
\end{cases}
\]

- **Simple function (RV):**

\[
\sum_{i=1}^{n} c_i I_{A_i}(\omega),
\]

where \(c_1, \ldots, c_n \in \mathbb{IR}\).

Note: can take \(\{A_i\}\) to be mutually disjoint.
Approximation with Simple Functions

- For any RV $X$ there exists a sequence of simple RV’s $X_n, n = 1, 2, \ldots$ such that $X_n(\omega) \to X(\omega)$ for all $\omega \in \Omega$

- Step 1: define

$$f_n(x) = nI_{\{x > n\}} + \sum_{k=0}^{n2^n-1} k2^{-n}I_{(k2^{-n},(k+1)2^{-n}]}(x)$$

- Step 2: if $X \geq 0$, set

$$X_n(\omega) = f_n(X(\omega))$$

- Step 3: in general, write $X = X_+ - X_-$ and set

$$X_n = f_n(X_+) - f_n(X_-)$$
Approximation with Simple Functions - illustration

\[ X(\omega) = \sum_{n=1}^{\infty} f_n(\omega) \]

- **n=1**
  - Graph showing \(X(\omega)\) for \(n=1\).
  - The function \(X(\omega)\) approximates the curve with a single step function.

- **n=2**
  - Graph showing \(X(\omega)\) for \(n=2\).
  - The approximation includes an additional step, improving the fit of the curve.

- **n=3**
  - Graph showing \(X(\omega)\) for \(n=3\).
  - The approximation further refines the fit with additional steps.

- **n=4**
  - Graph showing \(X(\omega)\) for \(n=4\).
  - The approximation closely follows the curve, demonstrating a higher level of accuracy.

Graphs illustrate the progressive approximation as \(n\) increases, showing how simple functions can approximate a more complex function.
Closure Properties of RV’s

Let \((\Omega, \mathcal{F})\) be a measurable space and let \(X_1, X_2, \ldots\) be a sequence of RV’s on it.

- If \(\lim_{n \to \infty} X_n(\omega)\) exists and is finite for all \(w \in \Omega\) then \(\lim_{n \to \infty} X_n\) is a RV
- If \(g : \mathbb{R}^n \to \mathbb{R}\) is Borel-measurable, then \(g(X_1, \ldots, X_n)\) is a random variable
- Special cases: the following are random variables
  - \(|X|
  - \sum_{i=1}^{n} \alpha_i X_i, \alpha_i \in \mathbb{R}
  - \prod_{i=1}^{n} X_i
  - \max(X_1, \ldots, X_n)\ and \ \min(X_1, \ldots, X_n)
  - \(X_+ \doteq \max(X, 0)\)\ and \(X_- \doteq -\min(X, 0)\)
σ-field generated by a RV

- The **σ-field generated by a RV** $X$, denoted $\sigma(X)$, is the smallest $\sigma$-field $\mathcal{G} (\subset \mathcal{F})$ for which $X$ is $\mathcal{G}$-measurable.

- Can show that

$$
\sigma(X) = \sigma(\{X \leq \alpha\}_{\alpha \in \mathbb{R}})
= \sigma(\{X \in B\}_{B \in \mathcal{B}})
$$

- If $X_1, \ldots, X_n$ are random variables on $(\Omega, \mathcal{F})$ then $\sigma(X_i, i = 1, \ldots, n)$ is the smallest $\sigma$-field containing $\sigma(X_i)$ for all $i = 1, \ldots, n$. 


Example: Exercise 1.2.9

- Consider a sequence of two coin tosses,
  \( \Omega = \{ HH, HT, TH, TT \} \), \( \mathcal{F} = 2^\Omega \)
- \( X_0 = 4 \)
- \( X_1 = 2X_0I_{\{\omega_1=H\}} + 0.5X_0I_{\{\omega_1=T\}} \)
- \( X_2 = 2X_1I_{\{\omega_2=H\}} + 0.5X_1I_{\{\omega_2=T\}} \)
- Find \( \sigma(X_0), \sigma(X_1), \sigma(X_2) \)
\(\sigma\text{-fields as Information}\)

- \(\sigma(X)\) contains the events \(A\) for which we can say whether \(\omega \in A\) or not, based solely on the value of \(X(\omega)\).
- A RV \(X\) is \(\mathcal{G}\)-measurable if and only if the information in \(\mathcal{G}\) is sufficient to determine the value of \(X\).
- A RV \(Y\) is \(\sigma(X_1,\ldots,X_n)\)-measurable if and only if \(Y = g(X_1,\ldots,X_n)\) for some Borel-measurable function \(g\).
Effects of Functions on Information

- If $X_1, \ldots, X_n$ are RV’s and $g$ is Borel-measurable, then
  \[ \sigma(g(X_1, \ldots, X_n)) \subseteq \sigma(X_1, \ldots, X_n) \]

- If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are RV’s defined on $(\Omega, \mathcal{F})$ such that
  \[ Y_k = g_k(X_1, \ldots, X_n) \quad \text{for each} \quad k = 1, \ldots, m \quad \text{and some Borel-measurable functions} \quad g_k, \quad \text{and} \]
  \[ X_i = h_i(Y_1, \ldots, Y_m) \quad \text{for each} \quad i = 1, \ldots, n \quad \text{and some Borel-measurable functions} \quad h_i, \]
  then
  \[ \sigma(X_1, \ldots, X_n) = \sigma(Y_1, \ldots, Y_m) \]