Last Time

- Continuous time Markov processes
- Transition probability functions
- Functions of Markov processes
- Strong Markov property

Today’s lecture: Section 6.2
Exponential Random Variables

• A RV $X$ has an exponential distribution with parameter $\lambda > 0$ ($X \sim \text{Exp}(\lambda)$) if for all $x \geq 0$,

$$\mathbb{P}(X > x) = e^{-\lambda x}$$

• *Memoryless property*: If $X$ has an exponential distribution then

$$\mathbb{P}(X > x + y | X > y) = \mathbb{P}(X > x) \text{ for all } x, y \geq 0$$

• If a continuous RV has the memoryless property, then the RV has an exponential distribution

• If $X_1, X_2, \ldots$ are i.i.d. Exp($\lambda$) then $T = X_1 + \cdots + X_n$ has a Gamma($n, \lambda$) distribution with density

$$f_T(t) = \frac{\lambda e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0$$
Poisson Process

- Let $X_1, X_2, \ldots$ be i.i.d. $\text{Exp}(\lambda)$ random variables
- Let $T_0 = 0$ and $T_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$
- A Poisson process with rate (or intensity) $\lambda$ is the continuous time stochastic process $\{N(t), t \geq 0\}$, where

$$N(t) = \sup\{n \geq 0 : T_n \leq t\} \text{ for } t \geq 0$$

- Interpretation:
  - $X_i$ are the times between occurrences of some event
  - $T_n$ is the time of the $n$th occurrence of the event
  - $N(t)$ counts the number of occurrences of the event in the time interval $[0, t]$
Illustration: Poisson Process Sample Path
Counting Process

A continuous time stochastic process $\{N(t), t \geq 0\}$ is a counting process if

- $N(0) = 0$
- Sample paths of $\{N(t), t \geq 0\}$ are piecewise constant
- Sample paths of $\{N(t), t \geq 0\}$ are nondecreasing
- Sample paths of $\{N(t), t \geq 0\}$ are right-continuous
- All jump discontinuities are of size one and there are infinitely many of them
- A counting process has state space $\mathbb{S} = \{0, 1, 2, \ldots\}$
Counting Process: Jump Times

Associated with each sample path of a counting process \( \{N(t), t \geq 0\} \) are the jump times \( 0 = T_0 < T_1 < T_2 < \cdots \), which satisfy

\[
T_n = \inf\{t \geq 0 : N(t) \geq n\}, \quad n = 0, 1, 2, \ldots,
\]

or equivalently

\[
N(t) = \sup\{n \geq 0 : T_n \leq t\}, \quad t \geq 0
\]

In particular, for any \( t \geq 0, n = 0, 1, 2, \ldots \)

\[
N(t) = n \text{ if and only if } T_n \leq t < T_{n+1}
\]
Basic Properties of Poisson Processes

Let \( \{N(t), t \geq 0\} \) be a Poisson process. Then:

- \( \{N(t), t \geq 0\} \) is a counting process
- \( N(t) < \infty \) a.s. for all \( t \geq 0 \) and \( N(t) \to \infty \) as \( t \to \infty \) a.s.
- For each \( t \geq 0 \), \( N(t) \) has a Poisson distribution with parameter \( \lambda t \), i.e.
  \[
  \mathbb{P}(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots
  \]
- For each \( t > s \geq 0 \), \( N(t) - N(s) \) has a Poisson distribution with parameter \( \lambda(t - s) \)
A *counting process* \( \{N(t), t \geq 0\} \) is a Poisson process if and only if

- \( \{N(t), t \geq 0\} \) has independent increments, and
- \( \{N(t), t \geq 0\} \) has stationary increments

Furthermore, there exists \( \lambda > 0 \) such that for all \( t > s \geq 0 \),

\[ N(t) - N(s) \text{ has a Poisson distribution with parameter } \lambda(t - s) \]
Poisson Process: Equivalent Definition 2

- Let \( \{N(t), t \geq 0\} \) be a counting process and let \( X_1, X_2, \ldots \) be the RV’s representing the times between jumps.
- \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \) if and only if \( X_1, X_2, \ldots \) are i.i.d. \( \text{Exp}(\lambda) \) random variables.
- In particular, if \( T_n = X_1 + \cdots + X_n \) then \( T_n \) has a Gamma distribution with parameters \( n \) and \( \lambda \).
Poisson Process: Equivalent Definition 3

A counting process \( \{N(t), t \geq 0\} \) is a Poisson process with rate \( \lambda \) if and only if

- \( \{N(t), t \geq 0\} \) is continuous in probability
- For any positive integer \( k \), \( 0 < t_1 < \cdots < t_k \), and nonnegative integers \( n_1, n_2, \ldots n_k \), and \( h > 0 \), both

\[
\begin{align*}
\mathbb{P}(N(t_k + h) - N(t_k) = 1 | N(t_j) = n_j, j \leq k) &= \lambda h + o(h), \\
\mathbb{P}(N(t_k + h) - N(t_k) \geq 2 | N(t_j) = n_j, j \leq k) &= o(h)
\end{align*}
\]

Notation: \( o(h) \) denotes a function \( g(h) \) which satisfies

\[
\frac{g(h)}{h} \to 0 \text{ as } h \downarrow 0
\]
Markov Property of Poisson Process

• Let \( \{N(t), t \geq 0\} \) be a Poisson process with rate \( \lambda \)
• Then \( \{N(t), t \geq 0\} \) is a homogenous Markov process with
• State space: \( S = \{0, 1, 2, \ldots\} \)
• Initial distribution: \( \pi(\{0\}) = 1 \)
• Stationary transition probability function:

\[
p_t(n + k|n) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad t \geq 0, n, k = 0, 1, 2, \ldots
\]

• A Poisson process is a strong Markov process
Compensated Poisson Process

- Let \( \{N(t), t \geq 0\} \) be a Poisson process with rate \( \lambda \)
- Let \( \{\mathcal{F}_t\} \) be the canonical filtration of \( N \)
- Define \( M(t) = N(t) - \lambda t \)
- The process \( \{M(t), t \geq 0\} \) is called a **compensated Poisson process**
- \( \{M(t), \mathcal{F}_t\} \) is a martingale
- \( \{M^2(t) - \lambda t, \mathcal{F}_t\} \) is a martingale