Last Time

- Doob decomposition (discrete time)
- Doob-Meyer decomposition (continuous time)
- Quadratic variation of stochastic process
- Quadratic variation of Brownian motion
- Martingale characterization of Brownian motion

Today’s lecture: Section 4.6
Branching Processes

• Let $N$ and $N_j^{(n)}$ be i.i.d. non-negative integer valued RV’s with finite mean $m = \mathbb{E}(N) < \infty$.

• A **Branching Process** is a discrete time SP $\{Z_n\}$ taking nonnegative integer values such that $Z_0 = 1$ and for any $n = 1, 2, \ldots$

$$Z_n = \sum_{j=1}^{Z_{n-1}} N_j^{(n)},$$

• We take $Z_{n+1} = 0$ if $Z_n = 0$
Interpretation

- $Z_n$ represents the size of the $n$-th generation of some population.
- $N_j^{(n)}$ represents the number of offspring of the $j$-th individual of the $(n-1)$-st generation.
- $m = \mathbb{E}(N)$ is the mean number of offspring for each individual.
Martingale Properties

• Let $\mathcal{F}_n = \sigma(N_j^{(k)}, k \leq n, j = 0, 1, 2, \ldots)$

• Then $\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = mZ_n$. In particular,
  ○ $\mathbb{E}(Z_n) = m^n$
  ○ If $m = 1$, $\{(Z_n, \mathcal{F}_n)\}$ is a martingale

• $\{(m^{-n}Z_n, \mathcal{F}_n)\}$ is a martingale
Probability of Extinction

- Let $p_{\text{ex}}$ denote the *probability of extinction*:
  
  $$p_{\text{ex}} = \mathbb{P}(Z_n = 0 \text{ for some } n \geq 0)$$

- Sub-critical process dies off:
  
  if $m < 1$ then $p_{\text{ex}} = 1$

- Critical process dies off:
  
  if $m = 1$ and $\mathbb{P}(N = 1) < 1$ then $p_{\text{ex}} = 1$

- Super-critical process can survive forever:
  
  if $m > 1$ then $p_{\text{ex}} < 1$
Probability of Extinction: Sub-critical Case

• Suppose $m < 1$ and let $X_n = m^{-n} Z_n$

• Since $\{X_n\}$ is a nonnegative martingale

• By the martingale convergence theorem, there exists a random variable $X_\infty$ such that

$$X_n \to X_\infty \text{ a.s. as } n \to \infty$$

• Since $X_\infty < \infty$ a.s. and $m < 1$, $Z_n \to 0$ a.s.

• Since $Z_n$ takes integer values, must have $Z_n = 0$ for some $n$, so $p_{ex} = 1$
Probability of Extinction: Critical Case

- Suppose \( m = 1 \) and \( IP(N = 1) < 1 \)
- Then \( \{Z_n\} \) is a martingale and as before there exists a random variable \( Z_\infty \) such that
  \[
  Z_n \to Z_\infty \text{ a.s. as } n \to \infty
  \]
- Proof of Proposition 4.6.5 shows that \( Z_\infty = 0 \) a.s. and \( p_{ex} = 1 \)
- Note that since \( IE(Z_n) = 1 \) for all \( n \), \( Z_n \) does not converge in \( L^1 \).
- Thus, \( \{Z_n\} \) is not uniformly integrable