Last Time

- Reflection principle for BM
- Brownian hitting times
- Running maximum of BM
- Law of large numbers for BM

Today’s lecture: Section 4.4.2, 4.5
Doob’s Submartingale Inequalities

• If \( \{X_t\} \) is a RCLL submartingale then for all \( x > 0 \) and \( t > 0 \)

\[
P\left( \sup_{0 \leq s \leq t} X_s > x \right) \leq \frac{IE|X_t|}{x}
\]

• If \( \{X_t\} \) is a martingale then for all \( x > 0 \) and \( t > 0 \)

\[
P\left( \sup_{0 \leq s \leq t} |X_s| > x \right) \leq \frac{IE|X_t|}{x}
\]

• Analogous versions exist for discrete time
Doob’s Maximal Inequality

- If either
  - \( \{X_t\} \) is a RCLL martingale, or
  - \( \{X_t\} \) is a nonnegative RCLL submartingale

- Then for all \( p > 1 \) and \( t \geq 0 \)

\[
\mathbb{E}\left[ \left( \sup_{0 \leq s \leq t} |X_s| \right)^p \right] \leq \left( \frac{p}{p - 1} \right)^p \mathbb{E}|X_t|^p
\]

- Analogous version exists for discrete time
Martingale Convergence Theorem

• If \( \{X_t, \mathcal{F}_t\} \) is a RCLL submartingale with

\[
\sup_{t \geq 0} \mathbb{E}(X_t^+) < \infty
\]

• Then there exists a RV \( X_\infty \) such that \( \mathbb{E}|X_\infty| < \infty \) and

\[
X_t \to X_\infty \text{ almost surely as } t \to \infty
\]

• In addition,

\[
\mathbb{E}|X_\infty| \leq \lim_{t \to \infty} \mathbb{E}|X_t| < \infty
\]
Illustration: Martingale Convergence

See Exercise 4.5.4
Martingale Convergence and Uniform Integrability

- If \( \{X_t, \mathcal{F}_t\} \) is a *uniformly integrable* RCLL submartingale with
  \[
  \sup_{t \geq 0} \mathbb{E}(X_t^+) < \infty
  \]

- Then there exists a RV \( X_\infty \) such that \( \mathbb{E}|X_\infty| < \infty \) and
  \[
  X_t \to X_\infty \text{ almost surely as } t \to \infty
  \]

- In addition,
  \[
  X_t \to X_\infty \text{ in } L^1 \text{ as } t \to \infty
  \]

- In this case, \( X_\infty \) is a *last element*, i.e.
  \[
  \mathbb{E}(X_\infty | \mathcal{F}_t) = X_t \text{ for all } t \geq 0
  \]
Uniformly Integrable Martingales

If \( \{X_t, \mathcal{F}_t\} \) is martingale then the following are equivalent:

- \( \{X_t\} \) is uniformly integrable
- \( \{X_t\} \) has a last element: i.e. there exists an \( \mathcal{F}_\infty \)-measurable RV \( X_\infty \) satisfying:
  - \( \mathbb{E}|X_\infty| < \infty \), and
  - \( \mathbb{E}(X_\infty|\mathcal{F}_t) = X_t \) for all \( t \geq 0 \)
- \( X_t \) converges almost surely as \( t \to \infty \)
- \( X_t \) converges in \( L^1 \) as \( t \to \infty \)
Levy Martingales

- Let $Y$ be a RV with $\mathbb{E}|Y| < \infty$ and let $\{\mathcal{F}_t\}$ be some filtration. Define $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$

- Then $\{\mathbb{E}(Y|\mathcal{F}_t), t \geq 0\}$ is a uniformly integrable martingale and as $t \to \infty$,

$$\mathbb{E}(Y|\mathcal{F}_t) \to \mathbb{E}(Y|\mathcal{F}_\infty),$$

almost surely and in $L^1$

- All uniformly integrable martingales are of this form