Correlated Conjunctions

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Introduction

A test for a conjunction, or a region that is activated in two or more independent SPMs, was proposed in [1, 5]. The test is based on Gaussian Random Fields (GRF) theory, and it approximates the probability of two zero-mean smooth isotropic Gaussian random fields simultaneously being above some level \( u \). In this paper, we extend the results to conjunctions where the component SPMs are possibly correlated, using some new results in GRF theory. The method is validated in a small simulation study.

Correlated Conjunctions

Formulation

The method used in [5] relies heavily on independence of the SPMs. In many situations this assumption is not satisfied. For instance, of the two SPMs, say \( Z_1(t) \) and \( Z_2(t) \) are test-statistics for a GLM testing for the effect of covariates \( X_1 \) and \( X_2 \) at every voxel, then the SPMs will be independent if, and only if
\[
(X^T X)^{1/2} = \mathbf{0}
\]
where \( X \) is the design matrix of the GLM. If the covariates \( X_1 \) and \( X_2 \) are not subject to experimental control than it is unlikely that the above condition will be satisfied.

Correlated Conjunctions & the GKF

The test for a conjunction, correlated or not, is based on the SPM
\[
Z(t) = \min_{i \in D} Z_i(t),
\]
specifically the maximum of \( Z(t) \) over the search region \( D \).

Some correlated conjunctions can be formulated in terms of the GKF (1). Specifically, suppose that the \( k \) isotropic SPMs \( Z_1(t), \ldots, Z_k(t) \) with covariances
\[
\text{Cov}(Z_i(t)) = \Sigma
\]
can be expressed as
\[
Z(t) = \sqrt{\Sigma} \cdot \tilde{c}(t)
\]
where \( \tilde{c}(t) = (c_1(t), \ldots, c_k(t)) \) is a vector if i.i.d. GRFs. Then, formal manipulations show that the expansion of \( Z(\tilde{c}(t)) \) of the SPM \( Z(t) = \min_{i \in D} Z_i(t) \) in \( S \) can be expressed as \( c^{-1/2} \cdot \text{K} \) for some finitely generated cone \( \text{K} \subset \mathbb{R}^k \) with non-zero vertex.

The GKF (1) implies that
\[
P \left( \inf_{t \in S} Z(t) \geq u \right) \geq \sum_{j \in \text{K}} \chi(\Sigma^{-1/2} j)^{-1} C_j(K).
\]

All that remains is to compute the coefficients \( C_j(K) \).

Conjunction of two Gaussian SPMs: \( k = 2 \)

The coefficients \( C_j(K) \) have a simple closed form expression only when \( k = 2 \). For \( k \geq 2 \), numerical integration can be used to compute \( C_j(K) \). When \( k = 2 \), we must be able to compute the probability that \( W \sim N(0, I_{2d}) \) is in the region depicted in Figure 2 below, which can be broken up into three regions. The details of the computation can be found in [2].

Simulation study

To validate the above theoretical results, we simulated 2500 realizations of smooth isotropic Gaussian noise with a FWHM of 20mm and compared the empirical (simulated) \( \alpha = 0.05 \) threshold to the threshold obtained using the GKF. The search region, for simplicity, was taken to be a cube of varying side length. As Figure 3 indicates, the thresholds predicted by the GKF are quite accurate over a range of box sizes and correlations.

Conclusion

In this paper, we describe how to use GRF theory to test for a correlated conjunction. While closed form formulas exist only for the conjunction of two correlated SPMs, the GRF formalism can be used to numerically obtain the EC densities for higher order conjunctions, though this will not be investigated here. Source code in R to compute thresholds for the conjunction of two correlated SPMs can be found at http://www-stat.stanford.edu/~jtaylor/correlconj.R

References