Digital Ridgelet Transform based on True Ridge Functions

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Abstract. We study a notion of ridgelet transform for arrays of digital data in which the analysis operator uses true ridge functions, as does the synthesis operator. There are fast algorithms for analysis, for synthesis, and for partial reconstruction. Associated with this is a transform which is a digital analog of the orthonormal ridgelet transform (but not orthonormal for finite n). In either approach, we get an overcomplete frame; the result of ridgelet transforming an $n \times n$ array is a $2n \times 2n$ array. The analysis operator is invertible on its range; the appropriately preconditioned operator has a tightly controlled spread of singular values. There is a near-parseval relationship.

Our construction exploits the recent development by Averbuch et al. (2001) of the Fast Slant Stack, a Radon transform for digital image data: it may be viewed as following a Fast Slant Stack with fast 2-d wavelet transform. A consequence of this construction is that it offers discrete objects (discrete ridgelets, discrete Radon transform, discrete Pseudopolar Fourier domain) which obey inter-relationships paralleling those in the continuum ridgelet theory (between ridgelets, Radon transform, and polar Fourier domain).

We make comparisons with other notions of ridgelet transform, and we investigate what we view as the key issue: the summability of the kernel underlying the constructed frame. The sparsity observed in our current implementation is not nearly as good as the sparsity of the underlying continuum theory, so there is room for substantial progress in future implementations.


1. Introduction

1.1. Ridgelets on the Continuum

Recently, several theoretical papers have called attention to the potential benefits of analyzing continuum objects $f(x, y)$ with $(x, y) \in \mathbb{R}^2$ using new bases/frames called ridgelets [3, 4, 12]
A ridge function \( \rho(x, y) = r(ax + by) \), that is to say, it is a function of two variables which is obtained as a scalar function \( r(t) \) of a synthetic scalar variable \( t = ax + by \) [20]. Geometrically, the level sets of such a function are lines \( ax + by = t \) and so the graph of such a function, viewed as a topographic surface, exhibits ridges. The function \( r(t) \) is the profile of the ridge function as one traverses the ridge orthogonally to its level sets.

In Candès’ thesis [3], a ridgelet is a function \( \rho_{a,b,\theta}(x, y) = \psi((\cos(\theta)x + \sin(\theta)y - b)/a)/a^{3/2} \) where \( \psi(t) \) is a wavelet – an oscillatory function obeying certain moment conditions and smoothness conditions. The continuous Ridgelet transform \( \mathcal{R}_f(a, b, \theta) = \langle f, \rho_{a,b,\theta} \rangle \) is defined on functions \( f \) in \( L^1 \) and extends by density to \( L^2 \). This transform obeys a Parseval relation and an exact reconstruction formula. Candès also showed that discrete decompositions were possible, so that for \( L^2 \) spaces of compactly supported functions one could develop a frame of ridgelets – a discrete family \( \{\varphi_{a_n, b_n, \theta_n}(x)\} \) serving the role of an approximating system.

The “classic ridgelets” of Candès are not in \( L^2(\mathbb{R}^2) \), being constant on lines \( t = x_1 \cos \theta + x_2 \sin \theta \) in the plane. This fact seems responsible for certain technical difficulties in the deployment and interpretation of discrete systems based on Candès’ notion of ridgelet. In [12] Donoho proposed to broaden the concept of ridgelet somewhat, allowing “wide-sense” ridgelets to be functions obeying certain localization properties in a radial frequency \( \times \) angular frequency domain. Under this broader conception, ridgelets no longer are of strict ridge the form \( \rho_{a,b,\theta}(x) \), so the elegant simplicity of formulation is lost. However, in exchange, it becomes possible to have an orthonormal set of “wide-sense” ridgelets. These orthonormal ridgelets are believed to be appropriate \( L^2 \)-substitutes for ridge functions, and to fulfill the goal of a constructive and stable system which although not based on true ridge functions are believed to play operationally the same role as ridge functions, compare [12, 13].

For either classic ridgelets or orthonormal ridgelets, the central issue is that such systems should behave very well at representing functions with linear singularities. As a prototype, consider the mutilated Gaussian:

\[
g(x_1, x_2) = 1_{\{x_2 > 0\}} e^{-x_1^2 - x_2^2}, \quad x \in \mathbb{R}^2. \tag{1.1.1}
\]

See Figure 1.

This is discontinuous along the line \( x_2 = 0 \) and smooth away from that line. Due to the singularity along the line, this function has coefficients of relatively slow decay in both wavelet and Fourier domains, so it requires large numbers of wavelets or sinusoids to represent accurately. The rate of convergence of best \( N \)-term superpositions of wavelets or sinusoids cannot be faster than \( O(N^{-1}) \). On the other hand, \( g \) can be represented by relatively few ridgelets: the rate of convergence of appropriate \( N \)-term superpositions of ridgelets or ortho-ridgelets can be faster than \( O(N^{-m}) \) for any \( m > 0 \). And the situation is the same for any rotation or translation of \( g \) so that the line \( x_2 = 0 \) becomes a line \( \cos(\theta)x + \sin(\theta)y = t \).
While perfectly straight singularities are rare, many two-dimensional objects concern imagery with edges, which may be regarded as curved singularities. While ridgelets per se do not provide the right tool for such curved singularities, Candès and Donoho have used ridgelets to construct a system called curvelets which gives high-quality asymptotic approximations to such singularities. Curvelets are ridgelets that have been dilated and translated and subjected to a special space/frequency localization explained in [6]. The rate of convergence of an appropriate $N$-term superpositions of curvelets is nearly $O(N^{-2})$ in squared error, whereas the comparable behavior for classical systems would by $O(N^{-1})$ or worse.

1.2. Discretization of Ridgelets

The conceptual attractiveness of this theoretical work drives us to consider the problem of translating it (if possible) from continuum concepts, useful in theoretical discussion, to algorithmic concepts capable of widespread application. It is initially by no means obvious how to do this or whether it can really be done. The theory of ridgelets is closely related to the theories of Radon transformation, and of rotation and scaling of images, all of which seem natural and simple on the continuum, and for which it is widely believed that there is no simple, inevitable definition for digital data.

A number of prior attempts at defining a digital ridgelet transform have been made; these will be discussed in detail further below.

In this paper, we propose a definition of digital ridgelet transform with several desirable properties. We believe that this definition is based on a clear understanding of the fundamental opportunities and limitations posed by data...
on a Cartesian grid, and has clear superiority over some other notions of discrete ridgelet transform which are, in our view, false starts.

Our definition offers:

- **Analysis and synthesis by true ridge functions.** The underlying analysis and synthesis functions depend on \((u, v)\) as \(\rho(u + bv)\) or \(\rho(v + bu)\). This means that the transform is geometrically faithful, and avoids wrap-around artifacts.

- **Exact reconstruction formula.** There is an iterative algorithm which in the limit gives exact reconstruction from the ridgelet transform.

- **Near-Parseval Relationship.** There is a variant of the DRT, which we call the (pseudo-) Ortho-Ridgelet Transform, in which the energy in coefficient space is equal to the energy in original space, to within a few percent.

- **Fast algorithm.** There is a fast algorithm requiring only \(O(N \log(N))\) flops for data sampled in an \(n\) by \(n\) grid, where \(N = n^2\) is the total number of data.

- **Continuum analogies.** The transform and related objects have structural relationships bearing a strong analogy with all the principal relationships that exist in the continuum case, between ridgelet transform, Radon transform, and Polar Fourier transform.

- **Cartesian data structures.** The transform takes data on a Cartesian grid and creates a rectangular coefficient array indexed according to a semi-direct product of simple integer indices measuring scale, location, and orientation.

- **Overcompleteness.** The transform takes an \(n\)-by-\(n\) array and expands it by a factor of 4 in creating the coefficient array.

We also compare properties of this DRT with its continuum counterpart, and with other discrete counterparts, particularly as regards sparse representation of objects with discontinuities along lines. We point out certain conceptual and practical advantages of the new transform, over, for example, the \(\mathbb{Z}_2^2\) transform proposed by Do and Vetterli [8], and certain advantages over straightforward discretizations of the Fourier plane proposed by Donoho [9] and Starck et al. [22].

Our current implementation provides a frame whose kernel does not have, in our view, sufficient sparsity to provide in the digital setting all the quantitative advantages offered by the continuum theory, leaving ample room for further improvements.
2. Digital Ridgelets

Let \( \psi_{j,k}(t) \equiv \psi_{j,k}(t; m) \) be the periodic discrete Meyer wavelet for the \( m \)-point discrete circle \(-m/2 \leq t < m/2 \) with indices \( J_0 \leq j < \log_2(m) \), and \( 0 \leq k < 2^j \); this is studied in, for example, Kolaczyk’s thesis [18]. This is actually defined as the discrete inverse Fourier transform

\[
\psi_{j,k}(t) = \sum_{w=-m/2}^{m/2-1} c_{j,k}^w \exp((i2\pi/m)wt)
\]

of a certain complex sequence \((c_{j,k}^w)\) which can be derived, e.g., using arguments in [1]. Since the formula makes sense for all \( t \) and not only for integers in the range \(-m/2 \leq t < m/2\), the periodic discrete Meyer wavelet is unambiguously defined not just at integral \( t \), but in fact for all real \( t \). Figure 2 displays a Meyer Wavelet of degree 2.

We will also have use for fractionally-differentiated Meyer wavelets, defined as follows. For a certain sequence \((\delta_w)\)

\[
\delta_w = \begin{cases} 
\sqrt{2w/m} & w \neq 0 \\
\sqrt{1/4m} & w = 0 
\end{cases}
\]

we apply this as a multiplier to the Fourier coefficients of \( \psi_{j,k} \), getting

\[
\tilde{\psi}_{j,k}(t) = \sum_{w=-m/2}^{m/2-1} \delta_w \cdot c_{j,k}^w \cdot \exp((i2\pi/m)wt).
\]

(Equivalently, we could define \( \tilde{\psi}_{j,k} = \Delta \ast \psi_{j,k} \), where \( \ast \) denotes \( m \)-point circular convolution and \( \Delta \) is the inverse discrete Fourier transform of \((\delta)\)). This is equally well viewed as a trigonometric polynomial defined at all \( t \). Figure 2 displays a fractionally-differentiated Meyer wavelet. For reasons that will be clear later, we also call the \( \tilde{\psi}_{j,k} \) normalized wavelets.

In this paper we consider images as \( n \) by \( n \) arrays indexed by coordinates \((u, v)\) ranging in the square \(-n/2 \leq u, v < n/2\) centered at \((0, 0)\). Let \( \theta_{\ell,n}^u \) be defined so that

\[
\tan(\theta_{\ell,n}^u) = 2\ell/n, \quad -n/2 \leq \ell < n/2; \quad \cotan(\theta_{\ell,n}^v) = 2\ell/n, \quad n/2 \leq \ell < n/2.
\]

The lines \( v = \tan(\theta_{\ell,n}^u)u + t \) we speak of as ‘basically horizontal lines’ and the lines \( u = \cotan(\theta_{\ell,n}^v)v + t \) we speak of as ‘basically vertical lines’. Each family of lines is equispaced in slope, rather than angle. Figure 3 illustrates this family of angles.
\begin{figure}
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{meyer_wavelet_deg2}
\caption{Meyer Wavelet of degree 2.}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{frac_deriv_meyer_wavelet_deg2}
\caption{Fractionally differenced Meyer wavelet of degree 2.}
\end{subfigure}
\caption{Left side: Meyer Wavelet of degree 2. Right side: Fractionally differenced Meyer wavelet of degree 2.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{frequency_space_lines}
\caption{Lines in frequency space corresponding to pseudopolar angles.}
\end{figure}

\textbf{Definition 1.} Let \( n \) be given. A digital ridgelet \( \rho_{j,k,s,t} \) is an \( n \) by \( n \) array built as ridge functions from Meyer wavelets by the formula
\[
\rho_{j,k,s,t}(u,v) = \psi_{j,k}(u + \tan(\theta^t_j)v), \quad s = 1,
\]
and
\[
\rho_{j,k,s,t}(u,v) = \psi_{j,k}(v + \cotan(\theta^t_j)u), \quad s = 2,
\]
where the parameter \( m \) underlying the definition obeys \( m = 2n \). We also call digital ridgelet any function built as ridge functions from fractionally-differentiated Meyer wavelets by the formula
\[
\tilde{\rho}_{j,k,s,t}(u,v) = \tilde{\psi}_{j,k}(u + \tan(\theta^t_j)v), \quad s = 1,
\]
and
\[
\tilde{\rho}_{j,k,s,t}(u,v) = \tilde{\psi}_{j,k}(v + \cotan(\theta^t_j)u), \quad s = 2.
\]
These definitions in fact guarantee that the resulting objects $\rho_{j,k,s,\ell}(u,v)$ and $\hat{\rho}_{j,k,s,\ell}(u,v)$ are digital samplings of true continuum ridge functions. We note that there are $m = 2n$ wavelets and $2 \times n$ angles $\theta_{\ell,n}$. For future use, we write $\lambda = (j,k,s,\ell)$ for quads occurring in this definition, and $\Lambda$ for the set of all $4n^2$ quads.

Figure 4 gives a few examples of such ridgelets.

**Figure 4.** Ridgelets.

**Definition 2.** The digital ridgelet analysis operator applied to an $n \times n$ image $(I(u,v) : -n/2 \leq u,v < n/2)$ is the array with $4n^2$ entries

$$RI = (\langle I, \rho_{\lambda} \rangle : \lambda \in \Lambda)$$

We also call digital ridgelet analysis the corresponding normalized operator

$$\tilde{RI} = (\langle I, \hat{\rho}_{\lambda} \rangle : \lambda \in \Lambda)$$

In either case, we conventionally think of the DRT as a $2n$ by $2n$ array, as in Figure 5, which shows the analysis of an object with linear singularity. Figure 6(a) gives a map of the coefficient space.

**Definition 3.** The digital ridgelet synthesis operator takes a $2n$ by $2n$ coefficient array $(\alpha_{\lambda} : \lambda \in \Lambda)$ into an $n \times n$ array

$$R^* \alpha = \sum_{\lambda} \alpha_{\lambda} \rho_{\lambda}.$$ 

We also call digital ridgelet synthesis the corresponding normalized operator

$$\tilde{R}^* \alpha = \sum_{\lambda} \alpha_{\lambda} \hat{\rho}_{\lambda}.$$ 

The notation $R^*$ is meant to suggest the adjoint operation, and in fact $R^*$ is precisely the formal adjoint of $R$.

The first main result is that these transforms are in a sense invertible, and so exact reconstruction is possible in principle.
Figure 5. Ridgelet analysis of an object with a linear singularity: Left side: Amplitude Map of Ridgelet coefficients. Right side: Amplitude Map of Ridgelet coefficients on a square root scale.

Figure 6. Map of coefficient space: Left side: Ridgelets, Right side: Ortho-Ridgelets.

**Theorem 1.** The operators $R$ and $R'$ are one-one and so invertible on their range.

The second main result is that these transforms are rapidly computable.

**Theorem 2.** The operators $R$, $R'$, $R^*$ and $R'^*$ can all be computed exactly in order $O(N \log(N))$ exact arithmetic operations, where $N = n^2$ is the number of entries in the $n \times n$ image.

The next 'result' is really a distillation of computational experience.

**Empirical Fact.** The normalized transforms $R$, and $R^*$ have their nonzero singular values within about 10% of each other. The generalized inverse $R^\dagger$
can be computed to seven digits accuracy in 4 iterations of a conjugate gradient solver.

As a corollary of this empirical result, we have that the system $\hat{\rho}_\lambda$ makes a frame, with the ratio of frame bounds empirically smaller than 1.10. Thus the system $(\hat{\rho}_\lambda)$ behaves nearly as well as would a tight frame or ortho basis.

It will also be important to consider a discrete analog of orthonormal ridgelets. Let $W_{i,\ell}(u)$ denote a discrete orthonormal Cohen-Daubechies-Feauveau-Jawerth boundary adjusted wavelet for the discrete interval $-n/2 \leq u < n/2$. There are $n$ of these wavelets, with indices $0 \leq i < \log_2(n)$ and $0 \leq \ell < 2^i$. For real sequences $(W_u)$ and $(V_u)$ indexed by $-n/2 \leq u < n/2$, let $\{W, V\}$ denote the inner product $\sum_u W_u V_u$.

**Definition 4.** Given the normalized discrete Ridgelet transform array $RI$, we define the (pseudo-) Ortho Ridgelet transform array $UI$ by taking the wavelet transform along the angular variable of $RI$:

$$(UI)_{j,k;i,\ell} = \{RI_{j,k;i,\ell}, W_{i,\ell}(\cdot)\}.$$  

**Figure 7.** Ortho-Ridgelet analysis of an object with a linear singularity: Left side: Amplitude map of Ortho-Ridgelet Coefficients. Right side: Ortho-Ridgelet Coefficients on a square root scale.

Figure 7 shows the ortho-ridgelet transform of the same Halfdome object as in Figure 5. It will be evident that the display is more sparse; the transform in $\theta$ has compressed the laterally elongated features in Figure 5 into more point-like features. Figure 6(b) gives a map of the coefficient space $M$ for this transform.

There is, of course, a Riesz representer for each coefficient of $UI$. For later use, let $\mu = (j, k; i, \ell)$ denote the tuple indexing $UI$, and let $\nu_\mu$ denote the Riesz representer of $(UI)_\mu$, i.e. the vector obeying

$$(UI)_\mu = \langle I, \nu_\mu \rangle.$$
Figure 8 gives an example of such a representer, which we will call a (pseudo-) ortho ridgelet. Just like the ortho ridgelets for the continuum, these are no longer true ridge functions; they crudely behave like fragments of ridgelets windowed by circular windows depending on i.

![Ortho-Ridgelets](image)

**Figure 8.** Some (pseudo-) Ortho-Ridgelets.

Now $U$ and $\tilde{R}$ are related by an orthonormal transformation of the space of $2n$ by $2n$ arrays, so it follows that all the injectivity and frame bounds properties obeyed by $\tilde{R}$ follow for $U$ as well.

**Corollary 1.** The $4n^2$ elements $(v_\mu : \mu \in M)$ make a frame with frame bounds empirically within about 10% of each other.

3. **Relation to Fast Slant Stack**

The article [2] defined the notion of *Fast Slant Stack* a certain kind of discrete Radon transform which is intimately connected to our notion of Ridgelet transform.

Given an array $I(u,v)$, a slope $a$ with $|a| \leq 1$, and an offset $z$, we initially define the Radon transform associated with the basically horizontal line $y = ax + b$ via

$$\text{Radon}(\{y = ax + b\}, I) = \sum_u \tilde{I}^z(u, au + b).$$

Thus, we are summing at $n$ values $(u, au + b)$ along the line $y = ax + b$. Since $a$ is not integral, the ordinates at which we are summing do not in general lie in the original pixel grid on integer pairs. Therefore, the values we are summing come
not from the original image $I$, but instead an interpolant $\tilde{I}^1(u, y)$, which takes integer values in the first argument, and real values in the second argument.

The interpolation “in $y$ only”, is performed as follows. Letting $m = 2n$, we define the Dirichlet kernel of order $m$ by

$$D_m(t) = \cotan(\pi t/m) \sin(\pi t)/m + i \sin(\pi t)/m.$$ 

We then set

$$\tilde{I}^1(u, y) = \sum_{v=-n/2}^{n/2-1} I(u, v) D_m(y - v).$$ 

We note that $D_m$ is an interpolating kernel, so that

$$I(u, v) = \tilde{I}^1(u, v), \quad -n/2 \leq u, v < n/2.$$ 

In the case of basically vertical lines, we define the Radon transform similarly, interchanging roles of $x$ and $y$:

$$\text{Radon}\{\{x = ay + b\}, I\} = \sum_u \tilde{I}^2(av + b, v),$$ 

with the interpolant defined analogously:

$$\tilde{P}^2(x, v) = \sum_{u=-n/2}^{n/2-1} I(u, v) D_m(x - u).$$ 

It is convenient to also have $\theta$ to represent the angle associated to the slope $s$.

**Definition 5.** The Slant Stack operator $S$ is defined by

$$(SI)(t, \theta) = \text{Radon}\{\{y = \tan(\theta)x + t\}, I\},$$

and for $\theta \in [\pi/4, 3\pi/4]$,

$$(SI)(t, \theta) = \text{Radon}\{\{x = \cotan(\theta)y + t\}, I\}.$$ 

where the intercept ranges through $-n \leq t < n$, and the angles vary through $\theta_{t,n}^{1} = \arctan(2\ell/n)$, $-n/2 \leq \ell < n/2$ and $\theta_{t,n}^{2} = \pi/4 + \arctan(2\ell/n)$.

We note that while $I$ is $n$ by $n$, $SI$ may be regarded as a $2n$ by $2n$ array. Figure 9 depicts the slant stack of the Half Dome image.

Recall the fractional differentiation operator introduced in Section 2. We can apply this to $t$-slices of $S$ to produce a normalized Slant Stack operator:

$$\tilde{SI}(\cdot; s, t) = \tilde{\Delta} \ast \tilde{SI}(\cdot; s, t)$$

The following properties of the Slant Stack operator have been established in [2].
Theorem 3. The operator $S$ is one-one and hence invertible on its range.

Theorem 4. The operators $S$ and $S^*$ can each be computed in order $O(N \log(N))$ flops.

Theorem 5. The operator $\tilde{S}$ has $n^2$ nonzero singular values; the ratio of the largest to smallest is bounded independently of $n$.

The next ‘result’ summarizes the computational experience in [2].

**Empirical Fact.** The normalized transform $\tilde{S}$, and adjoint $S^*$ have all their nonzero singular values within about 10% of each other. The generalized
inverse $\hat{S}^\dagger$ can be computed to seven digits accuracy in 4 iterations of a conjugate gradient solver.

We can now exhibit the relevance of the Slant Stack to our notion of Ridgelet transform. To do so, we let $\langle , \rangle$ denote the inner product purely in the $t$-variable.

**Theorem 6.** The Digital Ridgelet Transform is the 1-dimensional Meyer-wavelet transform, in $t$, of the Slant Stack Radon Transform:

$$(RI)(j, k, s, l) = \langle SI(\cdot; s, l), \psi_{j,k}(\cdot) \rangle.$$  

The normalized Digital Ridgelet Transform is the 1-dimensional Meyer-wavelet transform, in $t$, of the normalized Slant Stack Radon Transform:

$$(\hat{RI})(j, k, s, l) = \langle \hat{SI}(\cdot; s, l), \psi_{j,k}(\cdot) \rangle.$$  

With this equivalence established, it is clear that all the results of the last section follow immediately from the results quoted here. Since the Meyer wavelet transform is an isometry, we have

$$\|SI\|_2 = \|RI\|_2, \quad \|\hat{SI}\|_2 = \|\hat{RI}\|_2.$$  

All the norm bounds and norm ratios for the ridgelet transform and for the normalized Slant Stack transform are identical. Since the 1-dimensional Meyer transform costs $O(n \log(n))$ flops, and we perform this once for each column of $S$ the conversion from the $2n \times 2n$ slant stack domain to ridgelet domain costs a total of $O(n^2 \log(n))$ or $O(N \log(N))$ flops.

It remains to prove Theorem 6. Let $[.]$ denote the inner product in the $2n \times 2n$ slant domain, and let $1_{s_0} = 0$ denote the Kronecker sequence indexed by $(s, \ell)$ which has a 1 in position $s = s_0, \ell = l_0$ and is zero elsewhere. Then

$$\langle SI(\cdot; s, l), \psi_{j,k}(\cdot) \rangle = [SI, \psi_{j,k} \otimes 1_{s,l}].$$  

Now, by the definition of adjoint,

$$[SI, \psi_{j,k} \otimes 1_{s,l}] = (I, S^*(\psi_{j,k} \otimes 1_{s,l})).$$  

Finally, we arrive at the key observation:

$$S^*(\psi_{j,k} \otimes 1_{s,l}) = \rho_{j,k; s, l},$$  

a digital ridgelet is the (slant-) Radon backprojection of a wavelet living in a single $\theta$-slice. In turn, this follows because trigonometric interpolation is exact on Meyer wavelets,

$$\psi_{j,k}(x) = \sum_{t=-n}^{n-1} D_n(x - t)\psi_{j,k}(t).$$  

Indeed, $S^*$ involves application of the trigonometric interpolation operator to each column of $\psi_{j,k} \otimes 1_{s,l}$ and this gives exactly the same result at a given $(v = \tan(\theta) u + z)$ as applying the formula for $\psi_{j,k}$ directly at that point. Theorem 3.5 is established.
4. Structural Analogies

There are several analogies between the discrete ridgelet analysis proposed here and continuum ridgelet analysis. We believe that these analogies further support the correctness of our notion of digital Ridgelet analysis.

4.1. Two Continuum Radon Transforms

To understand our analogies it is important to introduce an important variant of the traditional continuum Radon transform – the continuum slant stack.

It is convenient in this paper to denote the Radon transform operator using the letter "X", as $R$ has already been taken by ridgelet transform, and $X$ suggests X-ray. Set

$$(Xf)(t, \theta) = \int f(x) \delta(x_1 \cos \theta + x_2 \sin \theta - t) \, dx,$$  \hfill (4.1.1)

where $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$.

Suppose we define, for $\theta \in [-\pi/4, \pi/4]$

$$Y^1 f(t, \theta) = \int f(x,y) \delta(t - x - y \tan(\theta)) \, dx \, dy,$$  \hfill (4.1.2)

and for $\theta \in [\pi/4, 3\pi/4]$

$$Y^2 f(t, \theta) = \int f(x,y) \delta(t - \cotan(\theta)x - y) \, dx \, dy$$  \hfill (4.1.3)

and encapsulate these in a single object $Yf$ defined by

$$Yf(t, \theta) = \begin{cases} Y^1 f(t, \theta) & \theta \in [-\pi/4, \pi/4] \\ Y^2 f(t, \theta) & \theta \in [\pi/4, 3\pi/4] \end{cases}.$$  

Then, if $f$ is a small pointlike ‘bump’, a display of $Yf$ will look like a broken line, with a break at the transition angle $\theta = \pi/4$.

The continuum Slant stack transform originated in seismics [7, 23]. Another field in which this continuum transform has been (independently) developed is medical tomography, where $Yf$ is called the Linogram [14, 15], in reference to the fact that points map under $Y$ into broken lines, whereas in the usual Radon transform, points map into sinusoids; because of this in medical tomography, the usual Radon transform is sometimes called the sinogram.

We remark that the continuum Slant stack and the continuum Radon transform contain the same information: for $\theta \in [-\pi/4, \pi/4]$

$$(Xf)(t \cdot \cos(\theta), \theta) = (Yf)(t, \theta);$$

a similar relationship holds for $\theta \in [\pi/4, 3\pi/4]$.
Digital Ridgelet Transform

It should be evident that the continuum slant stack has a close relationship to the discrete slant stack; hence, the above relationship between the continuum slant stack and the continuum Radon transform, provides a connection between the discrete slant stack and the continuum Radon, albeit with a certain amount of relabelling.

This observation is responsible for the several analogies described in this section.

4.2. Analogies between Polar and Pseudopolar Fourier Domains

To understand still better the underlying relationships, note that there are actually three important domains associated with traditional ridgelet analysis in the continuum case: the ridgelet domain, the Radon domain, and the so-called polar Fourier domain. To complete our understanding of the situation, we need to know about all three.

The polar Fourier transform is defined in terms of the usual Fourier transform by simple cartesian-to-polar transformation:

\[ \hat{F}(\omega, \theta) = \hat{f}(\omega \cos(\theta), \omega \sin(\theta)) \]

Through the well-known projection-slice theorem [7], many facts about the Radon domain can be translated isometrically into facts about the polar Fourier domain, and vice versa. The projection-slice theorem says that the one-dimensional Fourier transform in \( t \) of a fixed \( \theta \)-slice of the Radon transform yields precisely a slice of the Polar Fourier transform at the same \( \theta \). So, if \( f(x, y) \) is a function in \( L^2(\mathbb{R}^2) \) with Radon transform \( \mathcal{R}f(t, \theta) \) and polar Fourier transform \( \hat{F}(\omega, \theta) \) and if \( \mathcal{F}_1 \) denotes 1-dimensional Fourier transform in the first variable

\[ (\mathcal{F}_1, Xf)(\omega, \theta) = \hat{F}(\omega, \theta). \]  

(4.2.1)

The digital ridgelet transform obeys comparable relationships, between ridgelet domain, (slant-) Radon domain, and pseudopolar Fourier domain. The pseudopolar domain is discussed in detail in [2]. It offers a notion of polar Fourier domain better adapted to digital data. The digital Fourier domain is viewed as a sequence of squares, not circles, and the "radial shells" picked out by the (pseudo-) "radial" variable are squares.

Define the ordinary 2-d Fourier transform of \( I \) by

\[ \hat{I}(\xi_1, \xi_2) = \sum_{u,v} I(u, v) \exp \{-i(u\xi_1 + v\xi_2)\}. \]

The discrete pseudopolar Fourier transform \( P \) of the digital image \( I \) is defined for \(-n \leq k < n\) and \(-n/2 \leq \ell < n/2\) by sampling the \( 2-D \) Fourier transform at the collection of pseudopolar grid points illustrated Figure 11.

Formally,

\[ (PI)(k, s, \ell) = \begin{cases} \hat{I}(\pi k/n \tan(\theta_\ell^s), \pi k/n) & s = 1 \\ \hat{I}(\pi k/n \cot(\theta_\ell^s), \pi k/n) & s = 2 \end{cases} \]
The frequencies of evaluation in this relation do not lie in a Cartesian grid.

A discrete projection-slice theorem exists relating the (slant-) Radon transform and the pseudopolar Fourier transform. The discrete projection-slice theorem [2] says that if we take the 1-dimensional Fourier transform in $t$ of the slant-Radon data $SI(t, \theta)$, we get samples of the two-dimensional Fourier transform of $I$.

$$(\mathcal{F}_1 Sf)(\omega, \theta) = P(\omega, \theta),$$

where now $\mathcal{F}_1$ denotes discrete Fourier transform of length $2n$ in the first variable and $\omega = \pi k/n$, $-n \leq k < n$ and $\theta = \theta^\theta_{\ell n}$.

It follows that the (slant-) transform of digital data is isometric to the pseudopolar Fourier Transform.

We remark that this situation parallels a similar relationship for the continuum-domain Slant stack transform $Y f$ introduced above.

4.3. Analogies between Radon Isometries

An important consequence of (4.2.1) and (4.2.2): a simple postprocessing of the Radon transform creates an isometry. Indeed, as the Fourier transform is an isometry, we have that the norm of $f$ is conveniently measured in the polar Fourier domain by

$$||f||_2^2 = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{2\pi} |\tilde{F}(\omega, \theta)|^2 d\omega d\theta.$$
Indeed,
\[ \int_{-\infty}^{\infty} \int_{0}^{2\pi} |\tilde{F}(\omega, \theta)|^2 |\omega| d\omega d\theta = 2 \int_{0}^{\infty} \int_{0}^{2\pi} |f(r \cdot \cos(\theta), r \cdot \sin(\theta))|^2 r dr d\theta = 2 \|f\|_2 = 2 \|f\|_2. \]

It follows that the object
\[ \tilde{F}(\omega, \theta) = |\omega| \tilde{F}(\omega, \theta) \]
is isometric with \( f \), and equally that the new object
\[ \tilde{X}(\tau, \theta) = \mathcal{F}_1^{-1} \tilde{F}(\tau, \theta) \]
is a Radon-like object isometric with \( f \), which we call the Radon isometry. Now this object has an alternate definition. Let \( \overline{\Delta} \) denote the convolution operator on smooth \( L^2(\mathbb{R}) \) functions \( g \) defined on the Fourier side by the multiplier relation
\[ (\overline{\Delta} \ast g)(\omega) = |\omega| \hat{g}(\omega), \quad \omega \in \mathbb{R}. \]

It follows that
\[ \tilde{X}f(\tau, \theta) = \overline{\Delta} \ast Xf(\tau, \theta) \]
so that simple postprocessing of \( X \) by the Fourier multiplier \( |\omega| \) creates an isometry. Since \( \overline{\Delta} \ast \overline{\Delta} = -(\frac{d}{dt})^2 \), \( \overline{\Delta} \) is in an obvious sense a fractional differentiation operator.

The Radon isometry \( \tilde{X} \) bears a strong analogy with the normalized Slant Stack \( \tilde{S} \). Indeed, the fact that \( \tilde{X} \) is an isometry of \( L^2 \), makes it comparable to a matrix operator having all its singular values equal to one, while \( \tilde{S} \) has all its singular values within a reasonable percentage of each other. Moreover, \( \tilde{X} = (\overline{\Delta} \ast I)X \) i.e. a fractional differential operator is applied to the Radon transform; while \( \tilde{S} = (\Delta \ast I)S \) – a discrete analog of fractional operator is applied to the Slant stack.

### 4.4. Analogies between Ortho-Ridgelet Analyses

In the continuum case, the ridgelet orthobasis is built essentially as follows: we create an orthobasis of wavelets \( (W_\lambda(t, \theta)) \) living in the Radon domain \((t, \theta) \in \mathbb{R} \times [0, 2\pi]\). We then apply the inverse Radon Isometry \( \tilde{X}^{-1} \) to the wavelet basis, getting \( \rho_\lambda = \tilde{X}^{-1}W_\lambda \). In words, (ortho-) ridgelet analysis is wavelet analysis composed with Radon isometry.

In the discrete case, we do something entirely similar. We create an orthobasis of wavelets living in the (slant-) Radon domain \(-n \leq t < n, \theta_{1n}^\theta = \tan(2\ell/n), \theta_{2n}^\theta = \cotan(2\ell/n)\), by taking the direct product \( \psi_{j,k} \otimes w_{\ell,j} \). We then apply the normalized (slant-) operator \( S^j \) to this, getting (pseudo-) ortho-ridgelets \( \tau_{j,k;i,\ell} \).
4.5. Analogies Between Frequency-Domain Tilings

For additional insights, we can consider the ortho-ridgelets in the frequency domain rather than the Radon domain. The typical member \( \rho_\lambda(x_1, x_2) \) of the ortho-ridgelets basis for \( L^2(\mathbb{R}^2) \) can be defined in the frequency domain by

\[
\rho_\lambda(\xi) = |\xi|^{-\frac{1}{2}} (\psi_{j,k}(|\xi|) w_{i,\ell}^\xi(\theta) + \psi_{j,k}(-|\xi|) w_{i,\ell}^\xi(\theta + \pi) ) / 2 .
\]  \hspace{1cm} (4.5.1)

Here \( (\psi_{j,k}(t) : j \in \mathbb{Z}, k \in \mathbb{Z}) \) denotes an orthonormal basis of Meyer wavelets for \( L^2(\mathbb{R}) \), and \( (w_{i,\ell}^0(\theta), \ell = 0, \ldots, 2^k - 1; w_{i,\ell}^1(\theta), i \geq i_0, \ell = 0, \ldots, 2^i - 1) \) an orthonormal basis for \( L^2(0, 2\pi) \) made of periodized Lemarié scaling functions \( w_{i_0,\ell}^0 \) at level \( i_0 \) and periodized Meyer wavelets \( w_{i,\ell}^1 \) at levels \( i \geq i_0 \).

Now the 1-dimensional Fourier transform of a wavelet \( \hat{\psi}_{j,k}(\lambda) \) is a windowed sinusoid, with window supported in \( \lambda \in \left[ \frac{2\pi}{3} 2^j, \frac{8\pi}{3} 2^j \right] \). Hence, the Fourier transform of this ridgelet lives in a dyadic annulus of radius \( \approx 2^j \). The wavelet \( w_{i,\ell}^\xi(\theta) \) is mostly concentrated to a window near \( \theta_{i,\ell} = 2\pi \ell / 2^j \). Hence, the Fourier transform of the ortho ridgelet lives in an angular wedge. Combining these remarks, we have the pattern illustrated in Figure 12(a), which we call the ridgelet tiling of the frequency plane.

![Figure 12. Ridgelet tiling and Digital Ridgelet tiling.](image)

In the digital ridgelet case, we have something similar, but based on concentric squares rather than concentric circles, see Figure 12(b).

5. Example: HalfDome

We now use the HalfDome object to illustrate the relationships we have just discussed, and to underline the plausibility of ridgelet analysis.
Figure 9 shows the HalfDome object in Fourier space both with and without the Ridgelet tiling superposed. It is evident that the energy in the object is concentrated near a highly elongated sausage-shaped feature oriented at 90° to the orientation of the discontinuity in the HalfDome object in original space.

![2-d FFT of HalfDome on sqrt scale](image1)

![2-d FFT of HalfDome (sqrt scale) and Ridgelet Tiling](image2)

Figure 13. Fourier Transform of Halfdome on log scale. (a) without (b) with ridgelet tiling.

We of course hope that the ridgelet tiling is well-adapted to the underlying distribution of energy in the object under study - i.e. that only a few tiles are required to cover the bulk of the energy in the object and that only a few coefficients per tile are needed to represent the part of the object overlapping that tile. It is evident from the distribution of the energy in frequency space that there is a strong possibility that things will work out as hoped.

We now consider the situation in the Radon domain. Figure 14 reveals that the Slant Stack of the HalfDome object is smooth away from a singularity at a single \((s, l, t)\) value. Now, the ridgelet coefficients of HalfDome are roughly speaking such wavelet coefficients of the Slant Stack. It makes sense that a \(2 - d\) wavelet transform of this object will have sparse coefficients, with the large coefficients concentrated at indices associated with spatial positions near the singularity. Hence, we expect the ridgelet transform to be sparse.

However, to be rigorously correct, we must remember that the ridgelet coefficients arise from the wavelet transform of the normalized Slant Stack, which is portrayed in Figure 14.

The normalized Slant Stack exhibits - like the unnormalized Slant Stack - smoothness away from a singularity at a single point. Consequently, it is clear that we should indeed have sparsity of the 2-D wavelet transform and hence sparsity in the ridgelet domain, as indeed we observe in Figure 5. For extra clarity, we show the wavelet transform with nonlinear colormap transformation.
Continuing in our chain of equivalences, suppose we now take the discrete Fourier transform in $t$ of each $t$-slice of the normalized Slant Stack. Then we are viewing the (normalized) pseudopolar Fourier transform of HalfDome; this is displayed below.

In this pseudo-polar coordinate system, the object of interest is well-concentrated in a few tiles. Again, we see reason to expect a sparse representation.

6. **Sparsity of the Frame Kernel**

Both ridgelet transforms defined here are expansive; they transform an $n$ by $n$ array into a $2n$ by $2n$ coefficient array. Equivalently, each set of analysis and synthesis functions, having $4n^2$ elements for representing objects with $n^2$, is overcomplete.

As a result, it becomes important to study the frame kernel $G(\mu, \nu) = \langle \rho_\mu, \rho_\nu \rangle$, particularly as regards the sparsity of its rows kernel. If the ridgelet system were orthonormal, each row would have a single nonzero element on the diagonal, and hence be extremely sparse. In the overcomplete case, what we can hope for is that the nonzero elements in each row, once rearranged in order of decreasing amplitude, decay rapidly.

Owing to the obviously high inner products of true ridge functions oriented in neighboring directions, we should not expect rapid decay of the coefficients if we work in the ridgelet system, but should look instead to the ortho-ridgelet system.

To check the sparsity condition, we consider the following computational experiment: we synthesize a single ortho-ridgelet and then analyze that ridgelet, inspecting the resulting transform coefficients for rapid decay.
Digital Ridgelet Transform

Figure 15. Pseudopolar FT of HalfDome.

We give examples here in two cases.

6.1. Analysis of a Coarse-scale ridgelet

First, we consider an ortho-ridgelet in the main subband \( j = i \) at a coarse scale \( j \). Figure 16 shows the ortho-ridgelet, as well as the reconstruction of this ridgelet from 10 ortho-ridgelet terms.

Figure 16. Ortho-Ridgelet and 10-term Reconstruction.

Figure 17 shows the ortho-ridgelet synthesis plane – evidently a delta sequence peaking at the appropriate place in coefficient space. The ortho-ridgelet analysis plane is considerably more spread out than the synthesis plane. Figure 18 presents a close-up of the principal subband. The delta in the synthesis plane is replaced by a blob in the analysis plane.
Another display of behavior in the analysis plane simply plots the sizes of coefficients in decreasing order, as shown in Figure 19.

Evidently, there are few “big” coefficients, followed by a decaying tail.

6.2. Remarks on Decay

While the analysis plane is visually quite sparse, in fact the degree of sparsity exhibited by the transform is disappointing. This is caused by the fact – seen in Figure 19– that, after the initial quite rapid drop-off of coefficient amplitudes, there is a rather slow decay in the tail region. This sort of phenomenon is well known in the harmonic analysis of nonperiodic signals, and it is an interesting open question whether some variant on the procedure could repair this slow decay.
6.3. Edge Effects

It seems important to note that the kernel decay becomes worse the more the ridgelet in question is concentrated near the edges of the image. We consider a ridgelet at a fine scale that happens to lie near the corner of the image domain, and again show its ten-term reconstruction, see Figure 20.

![Rearranged Ridgelet Coefficients](image)

**Figure 19.** Decreasing Rearrangement of Ridgelet Analysis Plane.

![Dual Ridgelet](image) ![Reconstructed from 10 OR Coefficients](image)

**Figure 20.** Corner Ridgelet, and 10-term Reconstruction.

We again show the analysis and synthesis planes. The analysis plane is dramatically more spread out than before, exhibiting long stripes, see Figure 21. Figure 22 presents a close-up of the principal subband.

7. Comparisons

7.1. Comparison with $Z^2_p$-ridgelets

Do and Vetterli [8] have proposed a method of ridgelet analysis based on the use of the Radon transform for $Z^2_p$, the cartesian product of two copies of the integers
mod $p$, where $p$ is a prime. At a formal level, this has much to recommend it, including orthogonality. Essentially, one applies the $Z^2_p$ Radon transform, and then take the wavelet transform in the ‘t’ direction.

Unfortunately, the $Z^2_p$ Radon transform integrates over ‘lines’ which are defined algebraically rather than geometrically, and so the points in a ‘line’ can be rather arbitrarily and randomly spread out over the spatial domain. In consequence, the $Z^2_p$ ridgelets that are defined in this way have a rather strange appearance. Typical examples are shown on Figure 23.

Such ‘ridge functions’ have their support scattered haphazardly throughout the image plane, and resemble neither a a traditional ridge function, nor any spatially coherent object.

As a consequence of this behavior, partial reconstructions in the $Z^2_p$ system have errors which look very much like textured random noise. Figure 24 compares reconstruction of HalfDome by 50 $Z^2_p$ ridgelet coefficients with recon-
Figure 23. Examples of $Z_p^2$ Ridgelets.

Figure 24. Reconstruction of HalfDome by $Z_p^2$ ridgelets, and digital ridgelets.

Figure 25 considers an object used by Do and Vetterli [8] in their paper on $Z_p^2$ ridgelets, and compares reconstruction by 50 $Z_p^2$ ridgelet coefficients with reconstruction by 50 ridgelets based on the transform developed here. Again, the additional noisiness in $Z_p^2$ reconstruction is clear.

In summary, the $Z_p^2$ approach simply is not based on a geometrically faithful notion of ridgelet, and suffers from textural artifacts.
7.2. Comparison with earlier ridgelets

We briefly mention three other notions of ridgelet transform we know about.

In [9], an initial attempt was made to construct a discrete ridgelet transform operating in order $N \log(N)$ flops, where $N = n^2$ is the image size. The idea was based on approximate cartesian-to-polar resampling. One would overlay a true polar grid on the discrete Fourier transform, and approximately evaluate the discrete Fourier transform at polar grid points by interpolation from nearby cartesian grid points. It was shown that by this device one could obtain a Frame, provided the polar grid were sufficiently dense. However, this approach had three seeming drawbacks. First, it might be that it would require a very high degree of oversampling to obtain the Frame property. Second, the kind of interpolation required would involve considerable arithmetic for each desired polar grid point, with frequent accesses to data in a near a corresponding pixel location, which, on modern hierarchical memory computers might cause frequent cache misses, and correspondingly slow operations. Third, the method was indelicate to program, for example, because of attempting to ‘fit a round polar grid in a square cartesian box’ the were annoying special cases arose at the corners of the grid.

In [10], the strategy described in this paper, based on the pseudo-polar grid, was described and implemented, only with $m = n$ in the definition rather than the choice $m = 2n$ used here. A key advantage of this approach is that the underlying pseudopolar FFT requires only one-dimensional FFT’s and so is completely vectorizable; this is an advantage on modern hierarchical memory machines and machines with 1-dimensional FFT’s as built-in operations. The difference between the $m = n$ and $m = 2n$ versions comes in boundary behavior.
The \( m = n \) approach has ridgelets which wrap around at the border, while the \( m = 2n \) approach does not suffer from the wrap-around artifacts (see Figure 29).

![Figure 26](image)

**Figure 26.** Wrap around artifact of earlier ridgelets.

![Figure 27](image)

**Figure 27.** Comparison between Wavelet reconstructions, earlier Ridgelet reconstruction and OrthoRidgelet reconstruction of Half Dome.

The above cited papers were not released at the time they were written because of Stanford University’s patent filing in this area.

The paper [22], follows part of the strategy described in this paper, based on the pseudo-polar grid. However, instead of exact evaluation of the trigonometric polynomial \( \hat{I} \) on the pseudo-polar grid, it uses a simple nearest-neighbor interpolation scheme to evaluate pseudo-polar grid points in terms of nearby Cartesian grid points. The frame bounds available by this approach are considerably broader than those obtained by the exact interpolation used in this
paper, although for the image de-noising application described there, this factor does not seem to be very important.

Yoel Shkolnisky has pointed out another possible variation on our approach, discussed in [2] and forthcoming work. In extending the image we zero pad out to length \( m = 2n + 1 \) rather than \( 2n \), obtaining a corresponding expression for \( D_m \) which is purely real, namely

\[
D_m(t) = \frac{\sin(\pi t)}{m \cdot \sin(\pi t/m)}.
\]

This fixes a slightly inelegant property of the choice \( m = 2n \), namely that the ridgelets with the highest radial index \( j \) are not purely real – they have a small imaginary component, owing to the small imaginary component of the kernel \( D_{2n} \). With \( m = 2n + 1 \) this component goes away.

8. Discussion

We have described a notion of ridgelet transform which is able to synthesize or analyze using true ridge functions and which has various exact reconstruction and frame properties. It also obeys a series of relationships with a notion of digital Radon transform and a notion of digital polar Fourier transform which are precisely analogous to corresponding relationships that exist in the frequency domain. At its heart, the method is based on the use of pseudopolar FFT and Fast Slant Stack described in [2].

The principal disappointment of the existing implementation is the relatively slow decay of the ortho-ridgelet coefficients of a ortho-ridgelet. Figure 19 shows
Digital Ridgelet Transform


Figure 30. Left side: Decreasing Rearrangement of Ridgelet Analysis Plane, Right side: m-term approximation errors.

that, after an initially steep decline in coefficients, a kind of flat ‘background’ sets in. The slow decay is reminiscent of the behavior one would see from Gibbs phenomena in fourier analysis of discontinuities, or from critical sampling in Gabor analysis.

It is not hard to see why this behavior obtains, and to see that it is intrinsically tied to our central assumption – the use of ridge functions in a digital setting. Figure 32 illustrates the ridgelet analysis process; Figure 33 illustrates its adjoint, the ridgelet synthesis process. The ridgelet analysis process works as follows: an image is extended to twice its length, then sheared, then projected, then wavelet analyzed. The ridgelet synthesis process works ‘in reverse’: a delta sequence in the wavelet coefficient domain is inverted into a wavelet, which is then backprojected into a ridge function, which is then sheared into a tilted
ridge function, which is then mutilated.

This last step – mutilation – is the adjoint of extension by zero-padding, meaning that digital ridgelets, when viewed as an array, amount to a series of columns containing 1-d wavelets which have been brutally truncated from an $n \times 2n$ array to fit in an $n \times n$ array. If the wavelets were not mutilated, their inner products would decay rapidly with separation in index space; but the mutilation spoils the decay property.

From this point of view, it is rather obvious what to try next. One should develop analysis and synthesis transforms not based on ridge functions, but instead based on windowed ridge functions. We have not explored this proposal in detail here simply because although a straightforward and obvious extension, it violates the initial assumption marking the origin of this project: the use of true ridge functions.

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References
Figure 32. Stages of Radon analysis: padding, shearing, projection.

Figure 33. Stages of Radon synthesis (right to left): backprojection, shearing, mutilation.


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